Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 25, 2003*
Accepted: *March 17, 2003*

Comments on cosmological singularities in string theory

Micha Berkooz*

Department of Particle Physics, The Weizmann Institute of Science
Rehovot 76100, Israel
E-mail: Micha.Berkooz@weizmann.ac.il

Ben Craps, David Kutasov and Govindan Rajesh

Enrico Fermi Institute, University of Chicago
5640 S. Ellis Av., Chicago, IL 60637, USA
E-mail: craps@theory.uchicago.edu, kutasov@theory.uchicago.edu, rajesh@theory.uchicago.edu

Abstract: We compute string scattering amplitudes in an orbifold of Minkowski space by a boost, and show how certain divergences in the four point function are associated with graviton exchange near the singularity. These divergences reflect large tree-level backreaction of the gravitational field. Near the singularity, all excitations behave like massless fields on a 1+1 dimensional cylinder. For excitations that are chiral near the singularity, we show that divergences are avoided and that the backreaction is milder. We discuss the implications of this for some cosmological spacetimes. Finally, in order to gain some intuition about what happens when backreaction is taken into account, we study an open string rolling tachyon background as a toy model that shares some features with $\mathbb{R}^{1,1}/\mathbb{Z}$.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, Tachyon Condensation, Cosmology of Theories beyond the SM.

*Incumbent of the Recanati career development chair for energy research

JHEP03(2003)031

Contents

1. Introduction

Time dependent solutions in string theory are of interest, both for describing the early universe, and for studying various dynamical issues. In particular, the behavior near cosmological singularities has recently been studied by many researchers, including [1]–[28].

JHEP03(2003)031

The first step in such studies typically involves determining the wavefunctions of particles in the singular geometry, and in particular the manner in which these wavefunctions are to be continued through the singularities. This can be done e.g. by using orbifold [5, 8, 11] or coset conformal field theory [9, 11] techniques.

The second step involves an analysis of stability of the spacetime under small perturbations. Very general arguments lead one to believe that already classically a large backreaction of the geometry to small perturbations is to be expected, since any non-zero stress tensor gets infinitely amplified near a singularity [29]. Quantum mechanical effects generically lead to further large backreaction [30, 18].

The classical backreaction in a class of models which can be described near the singularity by certain lorentzian orbifolds has recently been studied in [8, 13, 16]. It was found that, as expected, these orbifolds suffer from instabilities associated with the divergent stress tensor of matter near the singularity. This is reflected in certain new divergences of the $2 \to 2$ tree level scattering amplitude of particles in the geometry. These divergences are associated with the region near the singularity, and signal the fact that, when backreaction is included, the curvature and string coupling in these backgrounds grow without bound (while without backreaction, the string coupling is weak everywhere, and the curvature vanishes everywhere except at the singularity).

We here will examine a different class of models, which reduce near a cosmological singularity to

$$\mathbb{R}^{1,1}/\mathbb{Z} \times \mathbb{R}^d \ , \tag{1.1}$$

where the orbifold generator acts as a boost in $\mathbb{R}^{1,1}$. This type of singularity, which is usually referred to in the literature as Milne or Misner spacetime, arises in a number of examples of recent interest:

1. The Nappi-Witten model [31, 9], in which this singularity appears at the intersection of two copies of a closed, big-bang/big-crunch universe, and certain non-compact static regions (the "whiskers" of [9]).

2. The big-bang/big-crunch cosmology described in [11], corresponding to a circle shrinking from finite to zero size, and then back to the original size.

3. The spacelike singularity of a BTZ black hole with $M > 0$, $J = 0$ [32, 33]; see [20] for a recent discussion and references.

4. $\mathbb{R}^{1,1}/\mathbb{Z}$ is also of independent interest, and has been studied as such in [5, 7] (in the latter paper with the Rindler wedges omitted).

The plan of the paper is as follows. In section 2 we review the structure of the orbifold (1.1), the wavefunctions on it, and the relation between the wavefunctions and the choice of vacuum for quantum fields on $\mathbb{R}^{1,1}/\mathbb{Z}$.

In section 3 we perform a string calculation of the tree level $2 \to 2$ scattering amplitude in the spacetime (1.1), and study its singularities. We are particularly interested in singularities of the scattering amplitude that are associated with the cosmological singularity (the fixed set of the boost action).

JHEP03(2003)031

In section 4 we compare the results of the string calculation to a gravity analysis of the same amplitude. This helps one identify the part of the amplitude associated with the backreaction to the growing stress tensor of perturbations near the orbifold singularity. We find that in general the dilaton remains finite near the singularity, while the curvature of the metric (including backreaction) blows up and causes the divergence found in section 3. We also find that one can fine-tune the initial conditions in such a way that the large backreaction is avoided.

The physical picture is the following. All quantum fields on (1.1) behave near the singularity like massless fields living on a $1+1$ dimensional cylinder. Left-moving excitations carry a large amount of T_{++}, while right-moving ones have a large T_{--}. Large backreaction occurs when both left and right movers are present. It is associated with processes in which left and right movers on the cylinder collide near the singularity. The fine-tuning referred to in the previous paragraph corresponds in this language to a situation where only left-movers or only right-movers are present near the singularity. One then has a wavefront moving with the speed of light to the left (or right) and no violent collisions/backreaction take place. General arguments suggest that in this case the solution is well behaved — e.g. α' and g_s corrections to the original background are small.

In section 5 we discuss some implications of the analysis of sections 3 and 4 to some of the systems mentioned earlier. In particular, we discuss the backreaction for different choices of vacuum in Milne spacetime. While in the vacuum inherited from the underlying Minkowski spacetime, the backreaction is always large, in another natural choice of vacuum one often finds a small backreaction. In the Nappi-Witten model, the backreaction to the modes studied in [9] is small. In the model of [11] one finds a large backreaction. We also briefly discuss the case of the non-rotating BTZ black hole; for a more comprehensive recent discussion see [34].

In situations where the classical backreaction is large, it would be interesting to understand what happens to the background when backreaction is taken into account. To gain insight into this interesting problem, we discuss in section 6 an open string toy model, which exhibits some of the features described for $\mathbb{R}^{1,1}/\mathbb{Z}$ above. The toy model is the dynamics of open strings on an unstable D-brane, in the background of a homogenous rolling tachyon [35].

We show that an effective field theory, which should provide a good description of the dynamics of the tachyon at late times [36], exhibits features similar to those found for the closed string systems in sections 3–5. Generic perturbations lead to "large backreaction" on the tachyon background at late times. By fine-tuning the solution in a way similar to that in section 4, one can arrange for the backreaction to be small. Moreover, by thinking of the D-brane as a collection of $D0$-branes, the fine-tuning in question is precisely that needed to keep the $D0$-branes at rest relative to each other at late times. The large backreaction in the general case is associated with $D0$-branes approaching each other and interacting in a non-trivial way via open strings stretched between them. We discuss the role of these interactions in the full open string problem, and the possible implications for the cosmological singularities studied in sections 3–5.

In section 7 we summarize the results, and discuss some open issues. Three appendices contain some useful technical results.

2. Wavefunctions and vertex operators in $\mathbb{R}^{1,1}/\mathbb{Z}$

2.1 Geometry of $\mathbb{R}^{1,1}/\mathbb{Z}$

In D-dimensional Minkowski space,

$$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \cdots + \left(dX^{D-1}\right)^2 , \tag{2.1}$$

define

$$\begin{aligned} X^{\pm} &= \frac{(X^0 \pm X^1)}{\sqrt{2}} ; \\ \vec{X} &= \left(X^2, \ldots, X^{D-1}\right) , \end{aligned} \tag{2.2}$$

so that the metric reads

$$ds^2 = -2dX^+ dX^- + d\vec{X}^2 . \tag{2.3}$$

$\mathbb{R}^{1,1}/\mathbb{Z}$ is obtained by orbifolding with the group $\mathbb{Z}$ generated by the boost

$$X^{\pm} \mapsto e^{\pm 2\pi} X^{\pm} . \tag{2.4}$$

The resulting spacetime consists[1] of four cones (times $\mathbb{R}^{D-2}$, which we will suppress in some of the formulae below), touching at the spacelike singularity $X^{\pm} = 0$. We will refer to the four cones as the "early time region" or "past Milne wedge" ($X^+, X^- < 0$), the "late time region" or "future Milne wedge" ($X^+, X^- > 0$), and the "regions with closed timelike curves" or "Rindler wedges" ($X^+X^- < 0$). In the early and late time regions, it is useful to define coordinates (t, x),

$$X^{\pm} = \frac{1}{\sqrt{2}} t e^{\pm x} , \tag{2.5}$$

in terms of which the metric and identification are

$$\begin{aligned} ds^2 &= -dt^2 + t^2 dx^2 ; \\ x &\sim x + 2\pi . \end{aligned} \tag{2.6}$$

It will also be convenient to define the conformal time coordinate η by (e.g. in the early time region)

$$t = -e^{\eta} , \tag{2.7}$$

so that the metric becomes

$$ds^2 = e^{2\eta} \left(-d\eta^2 + dx^2\right) . \tag{2.8}$$

[1] We neglect one dimensional pieces of spacetime that come from moding out the $X^+X^- = 0$ locus.

2.2 Wavefunctions

Wavefunctions on the orbifold (1.1) are wavefunctions on Minkowski space that are invariant under (2.4). A convenient basis of wavefunctions is given by [5]

$$\psi\left(X^+,X^-,\vec{X}\right)_{p^+,p^-;l} = \frac{e^{i\vec{p}\cdot\vec{X}}}{2\sqrt{2}\pi i}\int_{\mathbb{R}} dw\, e^{i(p^+X^-e^{-w}+p^-X^+e^{w}+lw)}\,, \qquad l\in\mathbb{Z}\,. \tag{2.9}$$

The normalization has been chosen such that the Klein-Gordon norm is 1. For $p^+,p^-\geq 0$ (which we will assume in what follows), these wavefunctions are superpositions of negative frequency plane waves in Minkowski space, so they describe excitations over the (adiabatic) vacuum inherited from Minkowski space (see e.g. [37]). The mass shell condition is $2p^+p^- = m^2$ (m being the two-dimensional mass, which includes contributions from momenta in any additional dimensions), so by shifting w and multiplying the wavefunction by a phase we can put

$$p^+ = p^- = \frac{m}{\sqrt{2}}\,. \tag{2.10}$$

Using the integral representation [38]

$$H^{(1)}_\nu(\tilde{z}) = \frac{1}{i\pi}e^{-\frac{1}{2}i\nu\pi}\int_0^\infty dy\, e^{\frac{1}{2}i\tilde{z}(y+\frac{1}{y})}\,y^{-\nu-1} \tag{2.11}$$

of the Hankel function $H^{(1)}_\nu$, valid for $0<\arg\tilde{z}<\pi$ or for $\arg\tilde{z}=0, -1<\mathrm{Re}\,\nu<1$, the wavefunction (2.9) can be brought to the form (for X^+, $X^-\geq 0$)

$$\begin{aligned}
\psi_{m,l} &= \frac{1}{2\sqrt{2}\pi i}\left(\frac{p^+X^-}{p^-X^+}\right)^{\frac{il}{2}}\int_0^\infty dy\, e^{i\sqrt{p^+p^-X^+X^-}(y+\frac{1}{y})}\,y^{il-1} \\
&= \frac{1}{2\sqrt{2}}e^{\frac{l\pi}{2}}\left(\frac{p^+X^-}{p^-X^+}\right)^{\frac{il}{2}}H^{(1)}_{-il}(\tilde{z}) \\
&= \frac{1}{2\sqrt{2}}e^{\frac{l\pi}{2}}\,e^{-ilx}H^{(1)}_{-il}(mt)\,,
\end{aligned} \tag{2.12}$$

where we have used

$$\tilde{z}\equiv 2\sqrt{p^+p^-X^+X^-} = mt\,. \tag{2.13}$$

We see that l is the momentum along the x circle (2.6). The expression of the wavefunction (2.9) in the other regions of $\mathbb{R}^{1,1}/\mathbb{Z}$ is described in appendix C.

The Hankel function $H^{(1)}_\nu(\tilde{z})$ can be written as

$$H^{(1)}_{-il}(\tilde{z}) = -\frac{1}{\sinh(l\pi)}\left(e^{-l\pi}J_{-il}(\tilde{z}) - J_{il}(\tilde{z})\right)\,, \tag{2.14}$$

where the Bessel function $J_{-il}(\tilde{z})$ has the power series expansion

$$\begin{aligned}
J_{-il}(\tilde{z}) &= \left(\frac{\tilde{z}}{2}\right)^{-il}\sum_{k=0}^\infty \frac{\left(-\frac{\tilde{z}^2}{4}\right)^k}{(k!)\Gamma(-il+k+1)} \\
&= \frac{1}{\Gamma(-il+1)}\left(\frac{\tilde{z}}{2}\right)^{-il}\sum_{k=0}^\infty \frac{\left(-\frac{\tilde{z}^2}{4}\right)^k}{(k!)(1-il)(2-il)\cdots(k-il)}\,.
\end{aligned} \tag{2.15}$$

Note that

$$e^{\frac{l\pi}{2}} H^{(1)}_{-il}(\tilde{z}) = e^{\frac{-l\pi}{2}} H^{(1)}_{il}(\tilde{z}) \,, \tag{2.16}$$

so that $\psi_{-l}(t,x) = \psi_l(t,-x)$; in particular, both are superpositions of negative frequency waves in Minkowski space. Their complex conjugates, which are superpositions of positive frequency modes in Minkowski space, can be written in terms of $H^{(2)}_{il} = (H^{(1)}_{-il})^*$. These positive frequency modes annihilate the vacuum inherited from Minkowski space.

Another basis of solutions to the wave equation is given by the functions (again in the region where X^+, $X^- \geq 0$)

$$\chi_{m,l}(t,x) = \frac{1}{2\sqrt{\sinh(\pi|l|)}} J_{i|l|}(mt) e^{-ilx} \tag{2.17}$$

together with their complex conjugates. For $t \to 0$ ($\eta \to -\infty$), (2.17) become purely negative frequency with respect to conformal time η, as can be seen using (2.15):

$$\chi_{m,l}(t,x) \sim \frac{1}{2\sqrt{\sinh(\pi|l|)}} \frac{1}{\Gamma(1+i|l|)} \left(\frac{m}{2}\right)^{i|l|} e^{i|l|\eta} e^{-ilx} \,. \tag{2.18}$$

As a consequence, the complex conjugate modes annihilate a state which in the limit $m = 0$ is called the conformal vacuum [37]. This is to be contrasted with the modes (2.12), which for $t \to 0$ involve both positive and negative frequencies with respect to η:

$$\psi_l \sim \frac{1}{2\sqrt{2}\sinh(\pi l)} \left[-\left(\frac{me^{\eta+x}}{2}\right)^{-il} \frac{e^{-\frac{\pi l}{2}}}{\Gamma(1-il)} + \left(\frac{me^{\eta-x}}{2}\right)^{il} \frac{e^{\frac{\pi l}{2}}}{\Gamma(1+il)} \right] . \tag{2.19}$$

Using the identity

$$|\Gamma(1+il)|^2 = \Gamma(1+il)\Gamma(1-il) = \frac{\pi l}{\sinh(\pi l)} \tag{2.20}$$

and writing $\Gamma(1+il) = e^{i\phi_l}\sqrt{\frac{\pi l}{\sinh(\pi l)}}$, with $\phi_l = -\phi_{-l}$, we can rewrite (2.19) as

$$\psi_l \sim \frac{1}{2\sqrt{2\pi l \sinh(\pi l)}} \left[-\left(\frac{me^{\eta+x}}{2}\right)^{-il} e^{-\frac{\pi l}{2}+i\phi_l} + \left(\frac{me^{\eta-x}}{2}\right)^{il} e^{\frac{\pi l}{2}-i\phi_l} \right] . \tag{2.21}$$

We see that, as mentioned in the introduction, wavefunctions of all particles behave near the Milne singularity like those of massless $1+1$ dimensional fields on the cylinder labelled by (η, x). We will return to this fact later.

Finally, one might want to consider wavefunctions which are Fourier transforms of (2.9):

$$\Psi(\sigma) = \sum_l \psi_l e^{-il\sigma} \,. \tag{2.22}$$

Since l is the momentum along the circle labelled by x (2.6), σ can be thought of as a position along this circle. Using (2.9), we have

$$\Psi(\sigma) = \sum_n e^{\frac{im}{\sqrt{2}}(X^- e^{-(\sigma+2\pi n)} + X^+ e^{\sigma+2\pi n})} \,, \tag{2.23}$$

so $\Psi(\sigma)$ is a plane wave in Minkowski space superposed with all its images under the orbifold group. As $\eta \to -\infty$, ψ_l, and hence Ψ, splits into left and right-moving parts, $\Psi = \Psi^- + \Psi^+$ where

$$\Psi^{\pm}(\sigma) = \sum_n a_n^{\pm}(\sigma) e^{in(x\pm\eta)} . \tag{2.24}$$

The coefficients $a_n^{\pm}(\sigma)$ can be extracted from (2.21):

$$a_n^{\pm}(\sigma) = \frac{\mp 1}{2\sqrt{2\pi n \sinh(\pi n)}} \left(\frac{m}{2}\right)^{\pm in} e^{\pm\frac{\pi n}{2} \mp i\phi_n + in\sigma} . \tag{2.25}$$

The Klein-Gordon norm of these wavefunctions diverges, so we will only consider (2.9) and (2.17) in this paper.

3. $2 \to 2$ scattering in string theory

We are interested in the behavior of strings near the singularity $X^{\pm} = 0$ of the orbifold (2.4). Therefore we will compute classical string scattering amplitudes involving the vertex operators (2.9), following [8] who studied a different time-dependent orbifold. We will focus on the $2 \to 2$ scattering amplitude, since this is the simplest case in which gravitational backreaction is expected to play a role.

Two and three point functions are studied in appendix A. It turns out that although the three point functions exhibit an interesting structure, this structure is associated to the asymptotic regions in the Milne wedges, not to the singularity. A very similar behavior was found in [8].

In the present section, we compute the four point function of the tachyon vertex operators (2.9). We will study the process $1 + 2 \to 3 + 4$. The corresponding amplitude is

$$\begin{aligned}
\langle \psi_3^* \psi_4^* \psi_1 \psi_2 \rangle &= \frac{1}{64\pi^4} \int_{-\infty}^{\infty} dw_1 \cdots dw_4 e^{i\sum \epsilon_i l_i w_i} \left\langle \prod_{i=1}^{4} e^{i(\frac{\epsilon_i m_i}{\sqrt{2}} X^- e^{-w_i} + \frac{\epsilon_i m_i}{\sqrt{2}} X^+ e^{w_i} + \epsilon_i \vec{p}_i \cdot \vec{X})} \right\rangle \\
&= \frac{(2\pi)^{20}}{4} \int dX^+ dX^- \int dw_1 \cdots dw_4 e^{i\sum \epsilon_i l_i w_i} e^{i\frac{X^-}{\sqrt{2}} \sum \epsilon_i m_i e^{-w_i} + i\frac{X^+}{\sqrt{2}} \sum \epsilon_i m_i e^{w_i}} \times \\
&\quad \times \delta^{24}\left(\sum \epsilon_i \vec{p}_i\right) \int d^2 z |z|^{p_1 \cdot p_3} |1 - z|^{p_1 \cdot p_4} .
\end{aligned} \tag{3.1}$$

The coefficients ϵ_i are 1 for the incoming particles 1 and 2, and -1 for the outgoing particles 3 and 4. The mass shell condition is (setting $\alpha' = 1$)

$$m^2 = -4 + \vec{p}^2 . \tag{3.2}$$

As before, m^2 is the effective two-dimensional mass squared, and we assume it to be positive, so we can define m to be positive as well. Define

$$v_i = e^{w_i - w_1} , \qquad i = 2, 3, 4 . \tag{3.3}$$

Then the Mandelstam invariants are given by

$$\begin{aligned} s &= -(p_1+p_2)^2 = -8 + m_1 m_2 \left(v_2 + \frac{1}{v_2}\right) - 2\vec{p}_1\cdot\vec{p}_2\,; \\ t &= -(p_1-p_3)^2 = -8 - m_1 m_3 \left(v_3 + \frac{1}{v_3}\right) + 2\vec{p}_1\cdot\vec{p}_3\,; \\ u &= -(p_1-p_4)^2 = -8 - m_1 m_4 \left(v_4 + \frac{1}{v_4}\right) + 2\vec{p}_1\cdot\vec{p}_4\,. \end{aligned} \tag{3.4}$$

We can reduce the expression for the four-point function to a single integral as follows. We first perform the z integral[2] in (3.1):

$$\int d^2z |z|^{p_1\cdot p_3}|1-z|^{p_1\cdot p_4} = 2\pi \frac{\Gamma(-1-\frac{s}{4})\Gamma(-1-\frac{t}{4})\Gamma(-1-\frac{u}{4})}{\Gamma(2+\frac{s}{4})\Gamma(2+\frac{t}{4})\Gamma(2+\frac{u}{4})}\,. \tag{3.5}$$

Defining $G(x) = \frac{\Gamma(-1-\frac{x}{4})}{\Gamma(2+\frac{x}{4})}$ and performing the w_1 and $X^\pm$ integrals, we can write the four point function as

$$\begin{aligned} &\frac{(2\pi)^{24}}{4}\delta^{24}\left(\sum \epsilon_i \vec{p}_i\right)\delta\left(\sum \epsilon_i l_i\right)\int_0^\infty dv_2 dv_3 dv_4\; G(s)G(t)G(u)\,\times \\ &\qquad \times\delta\left(m_1 + \sum_{j=2}^4 \epsilon_j m_j v_j\right)\delta\left(m_1 + \sum_{j=2}^4 \frac{\epsilon_j m_j}{v_j}\right)\prod_{j=2}^4 v_j^{i\epsilon_j l_j - 1}\,. \end{aligned} \tag{3.6}$$

We now perform the v_2, v_3 integrals. Setting the arguments of the delta functions to zero amounts to solving

$$\begin{aligned} m_1 + m_2 v_2 - m_3 v_3 - m_4 v_4 &= 0\,; \\ m_1 + \frac{m_2}{v_2} - \frac{m_3}{v_3} - \frac{m_4}{v_4} &= 0 \end{aligned} \tag{3.7}$$

for v_2, v_3. The solutions are

$$\begin{aligned} v_2 &= \frac{AB + m_2^2 - m_3^2 \mp \sqrt{\Delta}}{2m_2 B}\,; \\ v_3 &= -\frac{AB + m_3^2 - m_2^2 \pm \sqrt{\Delta}}{2m_3 B}\,, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} A &= -m_1 + m_4 v_4\,; \\ B &= -m_1 + \frac{m_4}{v_4}\,; \\ \Delta &= \left(m_2^2 - m_3^2\right)^2 - 2AB\left(m_2^2 + m_3^2\right) + A^2 B^2\,. \end{aligned} \tag{3.9}$$

[2] The z integral runs, as usual, over the (euclidean) worldsheet — the sphere or plane. This might seem like a problem since we are studying an inherently Minkowski signature spacetime, (1.1), and thus should take the worldsheet to have Minkowski signature as well. However, the euclidean calculation is only used here to arrive at the Shapiro-Virasoro amplitude, which can be taken to be the starting point of the analysis, and used directly in Minkowski spacetime.

Note that v_2, v_3 should be positive, so one should retain only the positive solutions among (3.8). Including a jacobian factor

$$\frac{1}{|m_2 m_3|}\frac{v_2^2 v_3^2}{|v_2^2 - v_3^2|} \tag{3.10}$$

from the delta functions and plugging in the solutions (3.8), the four point function can be reduced to the following single integral:

$$\sum \frac{(2\pi)^{24}}{4}\delta^{24}\left(\sum \epsilon_i \vec{p}_i\right)\delta\left(\sum \epsilon_i l_i\right)\int_0^\infty dv_4\, G(s)G(t)G(u)\frac{{v_2}^{il_2+1}{v_3}^{-il_3+1}v_4^{-il_4-1}}{|m_2 m_3({v_2}^2-{v_3}^2)|}\,, \tag{3.11}$$

where the sum runs over the positive solutions among (3.8). We are interested in the divergences of the four-point function, since they can potentially teach us something about the singularity.

Consider the $v_4 \to \infty$ region of the integral. In this limit the positive solution among (3.8) (corresponding to the upper sign) is given by

$$v_2 \approx \frac{m_4 v_4}{m_2}\,, \qquad v_3 \approx \frac{m_3}{m_1}\,, \tag{3.12}$$

so that the Mandelstam variables t and s are

$$t \approx -(\vec{p}_1 - \vec{p}_3)^2\,, \qquad \approx m_1 m_4 v_4\ . \tag{3.13}$$

This is the Regge limit $s \to \infty$, t fixed, in which

$$G(s)G(t)G(u) \to -\left(\frac{s}{4}\right)^{2+\frac{t}{2}}\frac{\Gamma[-1-\frac{t}{4}]}{\Gamma[2+\frac{t}{4}]}\,. \tag{3.14}$$

The $v_4 \to \infty$ limit of the four-point function is therefore

$$\begin{aligned}-\frac{(2\pi)^{24}}{2^{6-(\vec{p}_1-\vec{p}_3)^2}}\delta^{24}\left(\sum \epsilon_i\vec{p}_i\right)\delta\left(\sum\epsilon_i l_i\right)(m_1 m_4)^{1-\frac{1}{2}(\vec{p}_1-\vec{p}_3)^2}\left(\frac{m_4}{m_2}\right)^{il_2}\left(\frac{m_3}{m_1}\right)^{-il_3}\times\\ \times\frac{\Gamma[-1+\frac{(\vec{p}_1-\vec{p}_3)^2}{4}]}{\Gamma[2-\frac{(\vec{p}_1-\vec{p}_3)^2}{4}]}\int dv_4\, v_4^{-\frac{1}{2}(\vec{p}_1-\vec{p}_3)^2+i(l_2-l_4)}\,.\end{aligned} \tag{3.15}$$

The integral over v_4 diverges from $v_4 \to \infty$ whenever $(\vec{p}_1 - \vec{p}_3)^2 \leq 2$. The $v_4 \to 0$ limit is equivalent to the $v_4 \to \infty$ limit (they are complex conjugates).

In the above expression, we have set $\alpha' = 1$. For later comparison with the gravity calculation, it is also useful to consider the limit $\alpha' \to 0$, or more precisely the limit $\alpha' t \to 0$ ($\alpha' s$ and $\alpha' u$ cannot go to zero at the same time since $s+t+u = -16/\alpha'$; this is an irrelevant complication, which is due to the fact that the mass of the tachyon is of order the string scale. Similar results would be obtained for fields with masses well below the string scale). Near $x = 0$, we have $\Gamma[x-1] \sim -\frac{1}{x}$, so that the $\alpha' \to 0$ limit of the four-point function in the Regge limit becomes

$$\frac{(2\pi)^{24}}{16}\delta^{24}\left(\sum\epsilon_i\vec{p}_i\right)\delta\left(\sum\epsilon_i l_i\right)\left(\frac{m_1 m_4}{(\vec{p}_1-\vec{p}_3)^2}\right)\left(\frac{m_4}{m_2}\right)^{il_2}\left(\frac{m_3}{m_1}\right)^{-il_3}\int dv_4\, v_4^{i(l_2-l_4)}\,. \tag{3.16}$$

There are additional divergences from other regions of the integral (3.11). Some of them may be understood as different versions of the above. For example, $m_2^2 < m_3^2$, $m_1^2 < m_4^2$ and B a small positive number corresponds to large s, fixed u. In this regime the four-point function diverges whenever $(\vec{p}_1 - \vec{p}_4)^2 \leq 2$.

A different kind of divergence occurs when $v_2 = v_3$ and $(m_2 - m_3)^2 = (m_1 - m_4)^2$. This is an IR effect which is not associated with the singularity (see appendix B).

4. Gravity analysis

In this section we will show that the divergence (3.16) is due to exchange of gravitons near the singularity, and signals a large gravitational backreaction in that region. We will also see that in some situations the backreaction is milder than for generic kinematics, and the tree level $2 \to 2$ scattering amplitude is finite.

In subsection 4.1 we will compute the massless exchange contribution to the four point function (3.1) in an alternative way, and show that, in a certain kinematic regime, the dominant contribution to (3.16) comes from graviton exchange near the singularity. In subsection 4.2 we will study the backreaction in dilaton-gravity coupled to a scalar field, and relate these calculations to the scattering amplitudes studied earlier.

4.1 An alternative calculation of the $2 \to 2$ scattering amplitude in the $\alpha' t \to 0$ limit

In the $\alpha' t \to 0$ limit, with t a Mandelstam variable (3.4), (3.5) can be replaced by

$$2\pi \frac{\Gamma(-1-\frac{s}{4})\Gamma(-1-\frac{t}{4})\Gamma(-1-\frac{u}{4})}{\Gamma(2+\frac{s}{4})\Gamma(2+\frac{t}{4})\Gamma(2+\frac{u}{4})} \to -\frac{2\pi(p_1 \cdot p_2)^2}{t}\,. \tag{4.1}$$

In this limit, (3.5), and as a result the four point function studied in section 3, is dominated by a massless exchange in the t-channel. Thus, computing the $2 \to 2$ scattering of the wavefunctions (2.9) (with t small in string units) amounts to computing (3.1), with the last integral replaced by the r.h.s. of (4.1):

$$\begin{aligned}\langle \psi_3^* \psi_4^* \psi_1 \psi_2 \rangle = \frac{(2\pi)^{20}}{4} \int dX^+ dX^- \int dw_1 \cdots dw_4 e^{i\sum \epsilon_i l_i w_i} e^{i\frac{X^-}{\sqrt{2}}\sum \epsilon_i m_i e^{-w_i} + i\frac{X^+}{\sqrt{2}}\sum \epsilon_i m_i e^{w_i}} \times \\ \times \delta^{24}\left(\sum \epsilon_i \vec{p}_i\right)\left(-\frac{2\pi(p_1 \cdot p_2)^2}{t}\right).\end{aligned} \tag{4.2}$$

One way of computing (4.2) is to repeat the analysis of section 3, i.e. first perform the integrals over $X^\pm$, which give delta functions of momentum conservation on the covering space, and then integrate over w_i. The result is the $\alpha' t \to 0$ limit of (3.11), which reduces in the $v_4 \to \infty$ limit to (3.16). Note that in this limit one has $t \simeq -(\vec{p}_1 - \vec{p}_3)^2$ (see (3.13)). Also, to relate (4.2) to (3.16) one uses $s \approx m_1 m_2 v_2$ (see (3.12) and (3.13)); this amounts to $p_1 \cdot p_2 \approx -p_1^+ p_2^-$. These observations will be relevant for comparison with a second way of computing (4.2), to which we turn next.

Another way of computing (4.2) is to perform the w_i integrals first. Then one obtains

$$\langle\psi_3^*\psi_4^*\psi_1\psi_2\rangle = -(2\pi)^{24}\delta^{24}\left(\sum\epsilon_i\vec{p}_i\right)2\pi\int dX^+dX^-\,\partial_\mu\psi_{m_1,l_1}\partial_\nu\psi^*_{m_3,l_3}\frac{1}{\partial^2}\partial^\mu\psi_{m_2,l_2}\partial^\nu\psi^*_{m_4,l_4}\,. \tag{4.3}$$

The large v_4 divergences found in section 3 come from the term

$$\langle\psi_3^*\psi_4^*\psi_1\psi_2\rangle \simeq -(2\pi)^{24}\delta^{24}\left(\sum\epsilon_i\vec{p}_i\right)2\pi\int dX^+dX^-\,\partial_-\psi_{m_1,l_1}\partial_-\psi^*_{m_3,l_3}\frac{1}{\partial^2}\partial_+\psi_{m_2,l_2}\partial_+\psi^*_{m_4,l_4} \tag{4.4}$$

(see the previous paragraph). Now consider the contribution to (4.4) from the wedge $X^\pm > 0$; the other three wedges are discussed in detail in appendix C. We will see shortly that for a certain range of the transverse momenta, the integral (4.4) is dominated by the region of small X^+X^-. Let us assume for now that the integral (4.4) is dominated by the region of small X^+X^-, and that it is consistent to use the leading behavior (2.19) of the wavefunctions,

$$\psi_{m,l} \sim \frac{1}{2\sqrt{2}\sinh(\pi l)}\left[-\left(\frac{mX^+}{\sqrt{2}}\right)^{-il}\frac{e^{-\frac{\pi l}{2}}}{\Gamma(1-il)} + \left(\frac{mX^-}{\sqrt{2}}\right)^{il}\frac{e^{\frac{\pi l}{2}}}{\Gamma(1+il)}\right]. \tag{4.5}$$

Under these assumptions, the $1/\partial^2$ operator in (4.4) acts only on the $\exp(i\epsilon_j\vec{p}_j\cdot\vec{X})$ parts of the full wavefunctions. Thus, one has

$$\frac{1}{\partial^2} = -\frac{1}{(\vec{p}_2-\vec{p}_4)^2} \tag{4.6}$$

and (4.4) becomes

$$\frac{(2\pi)^{24}\delta^{24}(\sum\epsilon_i\vec{p}_i)2\pi\left(\frac{m_2}{\sqrt{2}}\right)^{-il_2}\left(\frac{m_4}{\sqrt{2}}\right)^{il_4}\left(\frac{m_1}{\sqrt{2}}\right)^{il_1}\left(\frac{m_3}{\sqrt{2}}\right)^{-il_3}e^{\frac{\pi}{2}(l_1+l_3-l_2-l_4)}l_1l_2l_3l_4}{64(\prod\sinh(\pi l_i))\Gamma(1-il_2)\Gamma(1+il_4)\Gamma(1+il_1)\Gamma(1-il_3)(\vec{p}_1-\vec{p}_3)^2}\times$$
$$\times\int_0^\infty\frac{dX^+dX^-}{(X^+X^-)^2}(X^+)^{i(l_4-l_2)}(X^-)^{i(l_1-l_3)}\,. \tag{4.7}$$

Using $X^\pm = \frac{1}{\sqrt{2}}e^{\eta\pm x}$, we can write the integral in the above expression as

$$\int_0^\infty dX^+\int_0^\infty dX^-(X^+)^{i(l_4-l_2)-2}(X^-)^{i(l_1-l_3)-2} =$$
$$= 2\int_{-\infty}^\infty d\eta\int_{-\infty}^\infty dx\left(\frac{e^\eta}{\sqrt{2}}\right)^{i(l_1+l_4-l_2-l_3)-2}e^{ix(l_3+l_4-l_1-l_2)}\,. \tag{4.8}$$

Performing the x integral, and defining $v = 2e^{-2\eta}$, we can simplify this to

$$2\pi\delta(l_1+l_2-l_3-l_4)\int_0^\infty dv\,v^{i(l_2+l_3-l_1-l_4)/2}\,. \tag{4.9}$$

In appendix C, we show that upon adding the contributions of the other three wedges of spacetime, (4.7) reproduces (3.16), with v playing the role of v_4. Thus, contributions from large v_4 (or large Mandelstam variable s with fixed t) correspond to contributions from the region near the singularity.

There is a small subtlety in the preceding discussion, which we would like to mention at this point. In (4.6), we assumed that in evaluating the $1/\partial^2$ operator, we can use the asymptotic form of the wavefunctions $\psi_{m,l}$ near the singularity $X^+X^- = 0$. In fact, if one is very close to the t-channel graviton pole, the exchanged particle can propagate for a large distance and the amplitude is *not* dominated by the behavior of the wavefunctions near the singularity. However, one can show that if the momentum transfer $(\vec{p}_2 - \vec{p}_4)^2$ is small compared to the string scale, but large compared to the two-dimensional masses squared, the approximation leading to (4.6) is valid at large v_4, and that subleading corrections in $m^2/(\vec{p}_2 - \vec{p}_4)^2$ correspond to subleading corrections in $1/v_4$ in the analysis of section 3.

Thus, we conclude that the divergence (3.16) of the four point function is associated with exchange of massless particles near the singularity. An interesting by-product of this analysis is that it points to particular kinematical situations where this divergence is absent. Consider two incoming particles 1 and 2 whose wavefunctions are purely left moving (or purely right moving) near the singularity, i.e. they only depend on X^+ (or only on X^-). Then it is clear that the contribution (4.4) responsible for the divergence vanishes. Examples of wavefunctions that are chiral near the singularity are given by (2.17). In section 5, we discuss some situations in which they are physically relevant.

4.2 Backreaction in dilaton gravity

In this subsection we would like to compute the classical backreaction of the graviton and dilaton to an incoming tachyon perturbation, and relate it to the amplitude calculations of the previous subsection.

Consider the action

$$S = \int d^D x \sqrt{-g} e^{-2\Phi} \left(R + 4 g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - g^{\mu\nu} \partial_\mu T \partial_\nu T - m^2 T^2 \right) , \qquad (4.10)$$

where $g_{\mu\nu}$ is the string frame metric, Φ the dilaton, and T a scalar field of mass m. It is convenient to transform to Einstein frame, by defining

$$\tilde{g}_{\mu\nu} = e^{-\frac{4\Phi}{D-2}} g_{\mu\nu} \, . \qquad (4.11)$$

The action is now

$$S = \int d^D x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{4}{D-2} \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \tilde{g}^{\mu\nu} \partial_\mu T \partial_\nu T - m^2 T^2 e^{\frac{4\Phi}{D-2}} \right) . \qquad (4.12)$$

In this action, Φ is essentially decoupled from $\tilde{g}$ and so we can treat them separately. The equation of motion for Φ, to leading order in Φ, is

$$\partial^2 \Phi = \frac{1}{2} m^2 T^2 \, . \qquad (4.13)$$

Thus, we see that the dilaton is only sensitive to the mass term, and in particular, it does not seem to diverge for the wavefunctions we consider in this paper (e.g., if the two T's on the r.h.s. of (4.13) are combinations of $(X^\pm)^{\mp i l_j}$, see (4.5)).

Now we turn to a discussion of the backreaction of the metric. We expand

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, . \qquad (4.14)$$

Plugging into the action (4.12) we get

$$S = \int d^D x \left\{ -\frac{1}{2} \left(\partial_\mu h^{\mu\nu} \partial_\nu h - \partial_\rho h^{\rho\sigma} \partial_\mu h^\mu_\sigma + \frac{1}{2} \partial_\mu h^{\rho\sigma} \partial^\mu h_{\rho\sigma} - \frac{1}{2} \partial_\mu h \partial^\mu h \right) + h^{\mu\nu} T_{\mu\nu} \right\}, \tag{4.15}$$

where Lorentz indices are raised and lowered with the flat metric $\eta_{\mu\nu}$ and we defined $h \equiv h^\mu{}_\mu$. The stress tensor that enters the action is

$$T_{\mu\nu} = \partial_\mu T \partial_\nu T - \frac{1}{2} \eta_{\mu\nu} \left[(\partial T)^2 + m^2 T^2 \right]. \tag{4.16}$$

We would like to fix the de Donder gauge

$$\partial_\mu h^{\mu\nu} = \frac{1}{2} \partial^\nu h\,. \tag{4.17}$$

We do this in a way analogous to the way Feynman gauge is introduced in QED. We add to the lagrangian a term

$$-\frac{1}{2} (\partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h)^2\,, \tag{4.18}$$

which leads to the following form of the action:

$$S = \int d^D x \left\{ -\frac{1}{4} \partial_\mu h^{\rho\sigma} \partial^\mu h_{\rho\sigma} + \frac{1}{8} \partial_\mu h \partial^\mu h + h^{\mu\nu} T_{\mu\nu} \right\}. \tag{4.19}$$

Varying with respect to $h_{\mu\nu}$, we find the equation of motion

$$\partial^2 h_{\mu\nu} = -2 \partial_\mu T \partial_\nu T - \frac{2}{D-2} m^2 T^2 \eta_{\mu\nu}\,. \tag{4.20}$$

It is useful in verifying this to write also the trace of this equation:

$$\partial^2 h = -2 (\partial T)^2 - \frac{2D}{D-2} m^2 T^2\,. \tag{4.21}$$

Note that this gives the Ricci tensor of the perturbed metric, since in the gauge (4.17)

$$R_{\mu\nu} = -\frac{1}{2} \partial^2 h_{\mu\nu} = \partial_\mu T \partial_\nu T + \frac{1}{D-2} \eta_{\mu\nu} m^2 T^2 \tag{4.22}$$

(see e.g. [39], eq. (6.6)). In particular, we see that there is large backreaction when the field T has both X^+ and X^- dependent pieces near the singularity. If the field T is chiral, say only a function of X^+, then only R_{++} will be large near the singularity, and the problem seems much milder behaved (for instance, powers of the Ricci scalar are finite, and α' corrections to the Einstein action constructed out of powers of the Ricci tensor can be neglected). This is in agreement with the observation made at the end of subsection 4.1, that divergences of four point functions are absent if the incoming fields are chiral.

To study the effect of the backreaction (4.13), (4.22) on the scalar field T, one classically integrates out $h_{\mu\nu}$, Φ, by plugging the solutions of their equations of motion back into the classical action (4.12). To order T^4, one finds

$$S_4 = \int d^D x \left\{ \partial_\mu T \partial_\nu T \frac{1}{\partial^2} \partial_\mu T \partial_\nu T - \frac{1}{2} \left[(\partial T)^2 + m^2 T^2 \right] \frac{1}{\partial^2} \left[(\partial T)^2 + m^2 T^2 \right] \right\}. \tag{4.23}$$

The second term on the r.h.s. of (4.23) is despite appearances non-singular. By integrating by parts, one can show that for on-shell tachyons it equals $-T^2\partial^2T^2/8$. The first term is of the form (4.3), which was used to compute the $2 \to 2$ scattering amplitude of T's in subsection 4.1.

5. Applications

The picture emerging from the analysis in sections 3 and 4 is that generic small perturbations of the Milne orbifold (1.1) lead to large classical gravitational backreaction, and that this is reflected in divergences in four point functions of these perturbations. The backreaction is milder if the perturbations are chiral near the singularity, i.e., if the incoming wavefunctions of particles 1 and 2 depend only on X^+ or only on X^- close to the singularity. The divergences in the four point function associated with the singularity are absent for such fine-tuned perturbations, as we mentioned at the end of subsection 4.1. In this section we will discuss some qualitative implications of this observation for a few spacetimes that look locally like the Milne orbifold (1.1).

5.1 Milne orbifold

When quantizing fields on the Milne orbifold, one has to choose a vacuum. One natural choice of vacuum is the one inherited from the Minkowski space prior to orbifolding. Excitations of this vacuum are described by the wavefunctions (2.12), which near the singularity involve both chiralities (see (2.19)). Thus, the analysis of sections 3 and 4 implies that this vacuum exhibits large classical backreaction to any perturbation.

On the other hand, as we discussed around (2.18), in another natural vacuum state (see e.g. [37]) the positive energy wavefunctions are given by (2.17). Near the singularity, they depend only on X^- for positive momentum l, and only on X^+ for negative momentum. So we conclude that in this state, there is large backreaction if one has incoming particles of both positive and negative momentum. Large backreaction is avoided if all incoming particles are moving in the same direction on the cylinder near the singularity (and similarly for outgoing particles).

5.2 Big crunch/big bang cosmology of [11]

In [11], an orbifold of a coset CFT was studied which describes a spacetime with a big crunch/big bang singularity, and with a number of asymptotic regions as well as compact regions with closed timelike curves (see also [40, 41]). Close to the big crunch/big bang singularity, the spacetime looks like a Milne orbifold (1.1). Natural in and out vacua were identified, and the amount of particle creation was computed in string theory. It was found that, due to the presence of different asymptotic regions and in particular the singularities connecting them, particle creation of any given mode did not decay with energy for large energies, unlike the situation in smooth spacetimes where it is know to decay exponentially with energy. This raises the suspicion that there should be large backreaction in this model.

The modes annihilating the natural incoming vacuum of [11] turn out to involve both chiralities near the Milne singularity, so there is large backreaction to any perturbation of

JHEP03(2003)031

this vacuum. It would be interesting to see if and how this is related to the large amount of particle creation in this model found in [11].

5.3 Nappi-Witten model

In [9], a coset CFT was studied which describes a four-dimensional spacetime containing a few copies of a closed big-bang/big-crunch universe, which was originally studied in [31], as well as a number of non-compact static regions which extend to spatial infinity ("whiskers"). The closed cosmological regions are attached to the whiskers at a singularity, which looks locally like the Milne orbifold (1.1).

As discussed in [9], it is natural to study scattering amplitudes of n to m particles in a given whisker. The incoming state corresponds to particles sent in from infinity in the whisker. Their wavefunctions are given by (3.29) in [9]; they have the property that near the Milne singularity (see figure 3 in [9]) they are chiral in region 1 (the whisker) and they vanish in the cosmological region I. Similarly, the wavefunctions of the outgoing states are chiral with the opposite chirality near the Milne singularity in region 1, and vanish in cosmological region II.

Thus, the analysis of sections 3 and 4 leads to the conclusion that the contribution from the whisker to $n \to m$ scattering amplitudes is finite (classically), while the contribution to these amplitudes from the closed cosmological regions vanishes. In fact, one can obtain these amplitudes by analytically continuing the correlation functions from euclidean space.

Actually, amplitudes in whisker 1 are not unitary in this case. The situation is analogous to that in eternal black hole spacetimes. The full geometry contains a second whisker (denoted by $1'$ in figure 3 of [9]), connected to the original one at the Milne singularity. Thus, there is another asymptotic region where information can go — spatial infinity in whisker $1'$. The situation is similar to that described in [42] for black holes (see also [20]). The full geometry contains two disconnected boundaries, at infinity in regions 1 and $1'$, on which asymptotic states are defined. Amplitudes in whisker 1 can be computed using a density matrix, corresponding to tracing out the degrees of freedom in whisker $1'$. One can also compute correlations between whiskers 1 and $1'$, which are non-zero because the states of the two whiskers are entangled.

The amplitudes of the modes described above do not get contributions from the cosmological big bang/big crunch regions. Thus, it is not very surprising that they do not give rise to large backreaction of the geometry. Generic incoming modes correspond to particles coming in from infinity in whisker 1, as well as from the cosmological region I. Such modes correspond to wavefunctions which contain both chiralities near the Milne singularity, and thus lead to large backreaction. More generally, amplitudes that probe dynamics in the compact, cosmological regions of spacetime are expected to suffer from the divergences discussed in sections 3 and 4, from one or both of its big bang and big crunch Milne singularities.

5.4 BTZ black hole

As mentioned in the introduction, non-rotating BTZ black holes have a spacelike singularity of Milne type, and one might wonder whether our analysis sheds any light on its fate in string theory.

JHEP03(2003)031

In asymptotically AdS spacetimes, one is interested in computing boundary correlations functions, which are the AdS analogues of S-matrix elements. Thus, the question is whether these boundary correlation functions are sensitive to physics near the BTZ singularity. A natural way of defining boundary correlation functions is by analytic continuation from euclidean space. Euclidean BTZ spacetime does not contain a singularity, and the boundary correlation functions on it are well behaved. One natural continuation from lorentzian to euclidean BTZ involves continuing "Schwarzschild time" $t_{\rm sch} \to i\theta$. This maps the euclidean black hole manifold to a single region outside the horizon of the black hole (say, region I in figure 1 of [42]). The euclidean correlation functions are mapped under this continuation to correlation functions[3] of insertions on the boundary of region I. This continuation is clearly insensitive to physics near the singularity, since only the behavior of wavefunctions in region I enters. Thus, our analysis is irrelevant for it.

Another continuation from Minkowski to euclidean BTZ involves a continuation of Kruskal time $t_{\rm kruskal} \to i\theta$. In this case, the euclidean black hole is mapped to the full extended lorentzian BTZ spacetime. Euclidean boundary correlation functions map to lorentzian correlation functions with insertions on both boundaries (in regions I and II). Assuming that these boundary correlation functions probe local physics in the full extended BTZ spacetime, and in particular near the singularity, they can be used to resolve the singularity; see [42] and [34] for recent discussions.

From the perspective of our discussion here, this second continuation is more puzzling. The wavefunctions (in the Hartle-Hawking state) that one gets in this continuation (e.g. eq. (2.7) in [42]) diverge near the BTZ singularity like

$$\phi \sim \log(X^+X^-) \tag{5.1}$$

in the coordinates used in sections 2–4. Thus, the contribution to these amplitudes from the vicinity of the singularity is divergent, as in the discussion of sections 3 and 4 above. This divergence signals a large backreaction of the metric, as in section 4. Since the full amplitude obtained by continuation from euclidean spacetime is finite, it must be that from the point of view of the discussion in sections 3 and 4, the divergences due to different singularities cancel [34]. However, this is a non-local cancellation, and there is some tension between the statement that it occurs in all correlation functions, and the expectation that boundary correlation functions can be used to probe local physics in the bulk of the full extended BTZ spacetime. Also, it is not completely clear in what sense one can neglect the backreaction near a particular singularity, which appears to be large. This issue clearly requires a better understanding.

6. Open string toy model

In this section we will discuss an open string model which shares some features with the closed string systems discussed in the previous sections. Consider a D-string in bosonic

[3] These correlation functions are not unitary since the CFT on the boundary of region I is entangled with one living on the boundary of region II, and the latter has been traced over in computing them.

string theory.[4] As is well known, the lowest lying excitation of the D-string is tachyonic. The condensation of this open string tachyon, T, leads to the disappearance of the brane. Condensation of spatially dependent modes of the tachyon leads to lower dimensional D-branes, which are also unstable and can further decay by tachyon condensation.

An interesting background of the theory on the D-brane is one in which the tachyon (which is taken to be constant along the D-string) starts at early time $x^0 \to -\infty$ at the top of its potential, $T = 0$, which corresponds to the original D-brane, and evolves at late time ($x^0 \to \infty$) to the bottom of its potential, $T = \infty$. The late time solution is not just the "no brane" (closed string vacuum) solution, since the energy of the original brane does not disperse in the classical approximation, but rather is stored in the kinetic energy of the open string field. This leads to a pressureless fluid "tachyon matter" state [35, 43].

In conformal field theory on the strip (i.e. classical open string theory), this "rolling tachyon" background is described by adding the boundary interaction

$$\delta S = \int d\tau e^{x^0(\tau)} \tag{6.1}$$

to the worldsheet lagrangian of a D-brane[5]. This boundary CFT has not been analyzed in detail (see [44] for a recent discussion), but it has been argued that the endpoint of the time evolution is highly unstable; for example, if one turns on an arbitrarily small coupling to closed strings, one expects the tachyon matter to decay into closed strings and disappear.

A closer analogue of the instabilities of cosmological spacetimes to small perturbations at early times discussed in previous section, would in this case be instabilities of the rolling tachyon background (6.1) to small open string perturbations at early times. We will next see that such instabilities indeed do arise and discuss their physical interpretation.

We will use a field theoretic effective action which has been argued [36] to give a good description of tachyon dynamics for large T. It is useful for our purposes, since this is the region which we are most interested in.

The action is

$$\begin{aligned} S &= \int dx dx^0 \, \mathcal{L}\,; \\ \mathcal{L} &= -V(T)\sqrt{1+\eta^{\mu\nu}\partial_\mu T \partial_\nu T}\,, \end{aligned} \tag{6.2}$$

where the potential is taken to be

$$V(T) = e^{-T}\,, \tag{6.3}$$

and the signature of the metric is $\eta = \mathrm{diag}(-,+)$. The potential (6.3) does not have a maximum unlike the actual potential of the tachyon field in open string theory. It can be thought of as describing the evolution of the tachyon for energies much smaller than

[4]A similar analysis can be performed for non-BPS branes and brane-antibrane systems in type-II string theory.

[5]More precisely, (6.1) is the form of the interaction at early times, $x^0 \to -\infty$, where the perturbation is small. The precise form of the perturbation at large x^0 depends on the renormalization prescription.

the energy of the original D-brane. The conclusions of our analysis would presumably be similar for other potentials which go exponentially to zero at large T, such as those that arise in boundary string field theory [45, 46, 47] (see [48, 49, 50] for discussions of more general potentials).

The energy density corresponding to (6.2) is

$$T_{00} = \frac{e^{-T}(1+T'^2)}{\sqrt{1-\dot{T}^2+T'^2}} . \tag{6.4}$$

In terms of the momentum conjugate to T,

$$\Pi(x) = \frac{\delta S}{\delta(\dot{T}(x))} = \frac{e^{-T}\dot{T}}{\sqrt{1-\dot{T}^2+T'^2}} , \tag{6.5}$$

the hamiltonian is

$$\begin{aligned} H &= \int dx\, T_{00} ; \\ T_{00} &= \sqrt{(\Pi^2+e^{-2T})(1+T'^2)} \end{aligned} \tag{6.6}$$

and the Hamilton equations of motion are

$$\begin{aligned} \dot{\Pi} &= \partial_x \left(\frac{T'\sqrt{\Pi^2+e^{-2T}}}{\sqrt{1+T'^2}} \right) + \frac{e^{-2T}\sqrt{1+T'^2}}{\sqrt{\Pi^2+e^{-2T}}} ; \\ \dot{T} &= \frac{\Pi\sqrt{1+T'^2}}{\sqrt{\Pi^2+e^{-2T}}} . \end{aligned} \tag{6.7}$$

A homogeneous solution of the equations of motion is given by

$$\begin{aligned} T_0 &= \log\left[\frac{1}{E}\cosh(x^0)\right] ; \\ \Pi_0 &= E\tanh(x^0) , \end{aligned} \tag{6.8}$$

which for late times approaches

$$\begin{aligned} T_0 &\sim x^0 ; \\ \Pi_0 &\sim E . \end{aligned} \tag{6.9}$$

Note, in particular, that the leading asymptotic behavior of T is independent of the energy density E, while that of Π does depend on E. More generally, there is a class of solutions of the equations of motion that goes for late times like [36]

$$\begin{aligned} T &\sim x^0 ; \\ \Pi &\sim f(x) , \end{aligned} \tag{6.10}$$

where $f(x)$ is an arbitrary function of the spatial coordinates, and the corrections to (6.10) are exponentially small at late times. Equation (6.6) implies that the energy density of such solutions is $T_{00} = |f(x)|$. In fact, it has been argued in [51] that the most general solution of the equations of motion (6.7) approaches at late times a solution of the first order equation

$$\dot{T}^2 - T'^2 = 1\,. \tag{6.11}$$

The solutions (6.10) are indeed of this form.

We would like next to perform a classical stability analysis of the homogenous rolling tachyon solution (6.8). Thus, we expand the tachyon field T as $T = T_0 + \phi$, where T_0 is the homogenous solution (6.8) and ϕ a small fluctuation (for early times). We would like to check whether ϕ remains a small perturbations as $t \to \infty$, or whether it grows to dominate the solution.

Expanding the action (6.2) to second order in ϕ, we find

$$\begin{aligned}\mathcal{L} = E\Bigg[&-\frac{1}{\cosh^2(x^0)} + \frac{1}{2}\left(\cosh^2(x^0)\dot{\phi}^2 - \phi'^2\right) + \left(\frac{\phi}{\cosh^2(x^0)} + \tanh(x^0)\dot{\phi}\right)\\ &- \left(\frac{\phi^2}{2\cosh^2(x^0)} + \tanh(x^0)\phi\dot{\phi}\right)\Bigg].\end{aligned} \tag{6.12}$$

The order ϕ^0 term is

$$\int dx^0 \frac{-E}{\cosh^2(x^0)} = -2E \tag{6.13}$$

(times the length of space, which we suppress — the problem is translation invariant in x). The linear term in ϕ, as well as the last term in brackets in (6.12) are total derivatives and can be neglected. The quadratic effective action for ϕ is thus [52]

$$S = \frac{E}{2}\int dx dx^0 \left(\cosh^2\left(x^0\right)\dot{\phi}^2 - \left(\vec{\nabla}\phi\right)^2\right), \tag{6.14}$$

which leads to the equations of motion

$$\frac{d}{dx^0}\left(\cosh^2(x^0)\dot{\phi}\right) - \phi'' = 0\,. \tag{6.15}$$

Substituting $u = \tanh(x^0)$, and $\phi(u, \vec{x}) = e^{ipx}F(u)$ the equation of motion simplifies to

$$\left(1 - u^2\right)F''(u) + p^2 F(u) = 0\,. \tag{6.16}$$

The solutions are hypergeometric functions:

$$\begin{aligned}F_1(u) &= F\left(\frac{-1-\sqrt{1+4p^2}}{4}, \frac{-1+\sqrt{1+4p^2}}{4}, \frac{1}{2}, u^2\right);\\ F_2(u) &= u\,F\left(\frac{1-\sqrt{1+4p^2}}{4}, \frac{1+\sqrt{1+4p^2}}{4}, \frac{3}{2}, u^2\right).\end{aligned} \tag{6.17}$$

We are particularly interested in the large time behavior of the solution (i.e. in the behavior as $u \to 1$). The two solutions approach constants,

$$F_1 \to C_1 = \frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{3+\sqrt{1+4p^2}}{4}\right)\Gamma\left(\frac{3-\sqrt{1+4p^2}}{4}\right)};$$
$$F_2 \to C_2 = \frac{\Gamma(\frac{3}{2})}{\Gamma\left(\frac{5+\sqrt{1+4p^2}}{4}\right)\Gamma\left(\frac{5-\sqrt{1+4p^2}}{4}\right)}. \tag{6.18}$$

C_1 and C_2 are generically non-zero, which implies that the approximation that ϕ is a small perturbation (in general) fails. Indeed, if F approaches a constant at late times, the second term in (6.14) makes an infinite contribution to the action, which overwhelms the 0'th order term (6.13), and one can check that the expansion of the square root that led from (6.2) to (6.12) is not justified in this case (at late times). Thus, generically, small perturbations create "large backreaction" to the original solution (6.8), much like in the discussion of sections 3 and 4. Another similar aspect of the two problems is that like there we can choose a linear combination of the two solutions (6.17), for which the "backreaction" is small. Indeed, the linear combination

$$F(u) = C_2F_1(u) - C_1F_2(u) \tag{6.19}$$

goes to zero exponentially at late times, and for it the quadratic action (6.14) is finite and the perturbative expansion (6.12) is well behaved. Thus, for roughly half of the possible initial states, we find mild backreaction.

All this is consistent with the general picture presented in [51] and in equation (6.11) above. For the fine-tuned initial data sets corresponding to (6.19), the late time behavior is of the form (6.10), with $f(x)$ determined by the initial conditions. This late time behavior corresponds to a small perturbation of the original solution (6.9). Generic initial conditions lead to a generic solution of (6.11). In particular, it is no longer true that the tachyon behaves like $T \sim x^0$ at late times, and the perturbation is not small there. Moreover, the authors of [51], who analyzed the general solutions of (6.11), showed that for generic initial conditions, the solutions develop caustics and become ill defined beyond a certain finite (but late) time, which depends on the initial conditions. This is another aspect that is reminiscent of the problem of backreaction near a cosmological singularity.

It is natural to ask what is the origin of the large backreaction for generic initial data, and what is the fate of the system once it is taken into account. We will next propose a possible physical picture which explains these phenomena, leaving a more complete understanding for future work.

Specifying generic initial data at early times corresponds to perturbing the solution (6.8) in a non-homogeneous way. In order to think about the time evolution of the perturbations, it is convenient to split the evolution into two steps.

JHEP03(2003)031

At a first step, the $D1$-brane decomposes into a set of $D0$-branes. For example, the tachyon profile[6]

$$T(x, x^0) = e^{\omega(p)x^0} \cos px \tag{6.20}$$

describes [54] a process in which the D-string decomposes into equidistant $D0$-branes at rest relative to each other. Each $D0$-brane has a tachyon living on it, and in the solution (6.20) these tachyons all grow uniformly. This is due to the high symmetry of the background (6.20); more generally, the $D0$-branes are located at arbitrary points x_i, have general relative velocities, and the tachyons on them are evolving in different ways.

The second step of the time evolution corresponds to the dynamics of the $D0$-branes. At this stage of the evolution, one expects the situation to depend on the relative motion of the $D0$-branes. If the $D0$-branes are at rest relative to each other, the late time dynamics should be well behaved. The $D0$-branes evolve independently from each other, with the tachyon on each growing as time goes by, but no collisions between different $D0$-branes taking place. This corresponds to solutions of the equations of motion of the effective action (6.2) with the late time behavior (6.10). The function $f(x)$ describes the density of $D0$-branes.

More generally, the $D0$-branes are in relative motion, and the late time dynamics is more complicated. In particular, when two $D0$-branes approach each other, the strings that connect them become light and can no longer be integrated out. We believe that this is the origin of the singularities associated with caustics found in [51], and the growth of fluctuations found earlier in this section. The effective action (6.2) appears to be well suited for describing the collective dynamics of well separated $D0$-branes; it breaks down when the interactions between different $D0$-branes become important at late times.

It should also be pointed out in this context that the analysis of caustics in [51] used the method of characteristics to study solutions of (6.11). In this method, the solution is obtained by following the behavior of certain worldlines of massive particles; caustics arise when different worldlines collide. The resulting picture is very reminiscent of what one would get by analyzing the motion of the $D0$-branes out of which the $D1$-brane is composed. It would be interesting to make the connection more precise.

Armed with a qualitative understanding of the origin of the singularities associated with generic perturbations of the solution (6.8), one can ask how these singularities are resolved in the full open string theory. As we saw, to do that one has to take into account the interaction between nearby $D0$-branes. In particular, the fact that the $D0$-branes can form bound states is not taken into account in the description (6.2). The analysis of [51] seems to suggest that for generic initial conditions, inhomogeneities in the tachyon field grow with time and give rise to localized clusters of $D0$-branes. These bounds states of $D0$-branes are quantum mechanical objects that need to be analyzed in the full $D0$-brane quantum mechanics. We will leave a more detailed analysis of this for future work.

Finally, one can ask what the open string example teaches us about closed string dynamics near cosmological singularities. Qualitatively, we see that the two problems are very similar. The effective action (6.2) seems to play a role similar to that played by

[6] This profile was recently studied in [53].

the dilaton gravity action (4.10) in the gravity analysis of section 4. In the closed string problem, the backreaction is mild when the incoming particles are all moving with the same velocity on the cylinder near the singularity. In the open string problem, the backreaction is mild when all the $D0$-branes which make up the D-string are moving with the same velocity at late times.

When $D0$-branes approach each other, one can no longer ignore effects due to open strings stretched between them. The analogous objects in the cosmological Milne singularity are twisted strings, which become light near the singularity and can no longer be neglected (see [26] for related comments). These twisted strings give rise to interactions between the incoming particles, the outcome of which has not been analyzed so far. A natural guess is that these interactions are a first step in the process of creation of black holes out of these colliding particles, as in [18]. This is similar to the creation of bound states of $D0$-branes in the time evolution of the tachyon in the open string problem. In fact, the analysis of the tachyon lagrangian (6.2) in [51] gives rise to a picture quite reminiscent of the time evolution of matter under the influence of gravity. The nonlinearity of the equations of motion has a similar effect in both cases: initial inhomogeneities are magnified, and the matter tends to cluster in different places in space.

7. Summary and discussion

We have computed string scattering amplitudes in the presence of a spacelike orbifold singularity (1.1), and found divergences similar to those of [8]. We argued that these divergences are due to graviton exchange near the singularity, and that they reflect large tree level gravitational backreaction. Interestingly, divergences can be avoided for special perturbations which behave like chiral two-dimensional fields near the singularity, and we discussed the extent to which such special perturbations are natural in some cosmological and black hole models. We also briefly discussed an open string rolling tachyon model, which seems to share some features with the cosmological backgrounds studied earlier, and might help understand the backreaction near the singularity.

It would be interesting to refine and extend this work in various directions, including:

1 We have argued that for fields that are chiral near the singularity, the gravitational backreaction is milder. Indeed, only the $(++)$ component of the Ricci tensor blows up, and four point functions are free of the usual divergences associated to the singularity. The same is expected to be true for higher point functions. It would be interesting to find an exact CFT description of the orbifold with chiral perturbations, and study its properties.

2 The system discussed in section 6 would be worth understanding in more detail. In particular, we proposed to think of the time evolution of inhomogeneous tachyon profiles in terms of the dynamics of a collection of D0-branes, and pointed out that this is very reminiscent of the worldlines of auxiliary massive particles used in [51] to solve the equations of motion of an effective field theory. It would be interesting

to try and make this analogy more precise, and understand the late time behavior of the system.

3 It would of course be interesting to understand the fate of the Milne singularity in cases where large backreaction occurs. We have made some qualitative observations at the end of section 6, and it would be nice if they could be made more precise.

4 It would be interesting to obtain a better understanding of the backreaction near the singularity of a BTZ black hole.

5 The analysis of this paper was entirely classical. It would be interesting to extend it to the quantum level, at least in cases where the classical backreaction is small.

Acknowledgments

We would like to thank P. Kraus, F. Larsen, H. Liu, E. Martinec, W. McElgin, G. Moore, S. Sethi and R. Wald for discussions and correspondence. B.C. would like to thank the Aspen Center for Physics and the Michigan Center for Theoretical Physics for hospitality while this work was in progress. D.K. thanks the Weizmann Institute and the Rutgers high energy theory group for hospitality. This work is supported in part by the Israel-U.S. Binational Science Foundation, the IRF Centers of Excellence program, the European RTN network HPRN-CT-2000-00122, the Minerva foundation, and by DOE grant DE-FG02-90ER40560 and NSF grant PHY-9901194.

A. Two and three-point functions in $\mathbb{R}^{1,1}/\mathbb{Z}$

In the text, we focused on a particular divergence of the four point function (3.1) of the operators (2.9), because, as we have argued, this divergence is directly associated with the singularity of the Milne orbifold. However, there are other divergences in three and four point functions of (2.9), which can be interpreted as infrared divergences. For completeness, we discuss two and three point functions in the present appendix, and infrared divergences in four point functions in appendix B.

A.1 Two point function

The two point function of two wavefunctions (2.9) is computed as follows:

$$\langle \psi_1^* \psi_2 \rangle = \frac{1}{2} \int dw_1\, dw_2\, e^{i(-l_1 w_1 + l_2 w_2)}\, \delta(-p_1^+ e^{-w_1} + p_2^+ e^{-w_2}) \delta(-p_1^- e^{w_1} + p_2^- e^{w_2})\,. \quad \text{(A.1)}$$

We assume $p_1^+, p_1^-, p_2^+, p_2^- > 0$. Using (2.10), and writing $w_1 = w_+ + w,\ w_2 = w_+ - w$, this becomes

$$2 \int dw_+\, e^{i(-l_1+l_2)w_+} \int dw\, e^{-i(l_1+l_2)w}\, \delta(-m_1 e^{-w} + m_2 e^{w}) \delta(-m_1 e^{w} + m_2 e^{-w})\,. \quad \text{(A.2)}$$

The first integral gives $2\pi\delta_{l_1,l_2}$ if we only integrate over values of w_+ that cannot be identified by the action of (2.4) on the wavefunctions. (Otherwise it would give $2\pi\delta(l_1-l_2)$.) We finally obtain

$$\langle\psi_1^*\psi_2\rangle = \frac{2\pi}{(m_1+m_2)}\delta_{l_1,l_2}\delta(m_1-m_2)\,. \tag{A.3}$$

We now would like to know which region of spacetime gives the dominant contribution to (A.3). Rewrite (A.1) as

$$\langle\psi_1^*\psi_2\rangle = \frac{1}{8\pi^2}\times \tag{A.4}$$
$$\times\int dX^+dX^-\int dw_1\,dw_2\,e^{i(-l_1w_1+l_2w_2)}\,e^{iX^-(-p_1^+e^{-w_1}+p_2^+e^{-w_2})}e^{iX^+(-p_1^-e^{w_1}+p_2^-e^{w_2})}\,.$$

We first perform the w_1, w_2 integrals:

$$\langle\psi_1^*\psi_2\rangle = \frac{1}{8}e^{(l_1+l_2)\frac{\pi}{2}}\int dX^+dX^-\left(\frac{p_1^+X^-}{p_1^-X^+}\right)^{-\frac{1}{2}il_1}\left(\frac{p_2^+X^-}{p_2^-X^+}\right)^{\frac{1}{2}il_2}(H^{(1)}_{-il_1}(\tilde{z}_1))^*H^{(1)}_{-il_2}(\tilde{z}_2)\,, \tag{A.5}$$

where, as in (2.13), $\tilde{z}_i = 2\sqrt{p_i^+p_i^-X^+X^-}$. (The expression (A.5) is accurate in the region $X^\pm>0$.) We now claim that the result (A.3) comes from the asymptotic regions $X^+X^-\to\infty$ of the past and future Milne wedges. In order to argue for this, we first concentrate on the future Milne wedge $X^\pm>0$ and replace the Hankel functions in (A.5) by their asymptotic expressions for large values of the argument: $H_\nu(z)\sim\sqrt{\frac{2}{\pi z}}e^{i(z-\pi\nu/2-\pi/4)}$. The justification for our claim will be that this procedure reproduces the correct result (A.3). Using (2.5), (2.10) and (2.13), the contribution of the future wedge to (A.5) becomes

$$\frac{1}{4\pi\sqrt{m_1m_2}}\int dx\,e^{-ix(l_2-l_1)}\int_0^\infty dt\,e^{-i(m_1-m_2)t}\,. \tag{A.6}$$

Adding the analogous contribution from the past Milne wedge amounts to extending the range of the t integral to the whole real line. The result is

$$\frac{\pi}{\sqrt{m_1m_2}}\delta_{l_1,l_2}\delta(m_1-m_2) = \frac{2\pi}{(m_1+m_2)}\delta_{l_1,l_2}\delta(m_1-m_2)\,, \tag{A.7}$$

in agreement with (A.3). We conclude that the dominant contribution to the two-point function comes from the $t\to\pm\infty$ asymptotics of the wavefunctions in the Milne wedges. This is in agreement with the fact that the wavefunctions (2.9) decay exponentially in the Rindler wedges. The result of this appendix should be contrasted with that of section 4 and appendix C, where a divergence is analyzed that comes from the singularity. In that computation, the dominant contribution comes from the region of spacetime near the singularity, and the four wedges make a comparable contribution.

A.2 Three point function

The three point function reads

$$\langle\psi_1^*\psi_2\psi_3\rangle = \frac{1}{2\sqrt{2\pi}i}\int dw_1\,dw_2\,dw_3\Big(e^{i(-l_1w_1+l_2w_2+l_3w_3)}\delta(-m_1e^{-w_1}+m_2e^{-w_2}+m_3e^{-w_3})\times$$
$$\times\,\delta(-m_1e^{w_1}+m_2e^{w_2}+m_3e^{w_3})\Big)\,. \tag{A.8}$$

where we have used (2.10). The w_1 integral gives

$$\frac{1}{2\sqrt{2}\pi i}\int dw_2\,dw_3\left(e^{i(l_2w_2+l_3w_3)}m_1^{-il_1}(m_2e^{-w_2}+m_3e^{-w_3})^{-1+il_1}\times\right.$$
$$\left.\times\,\delta\left(-\frac{m_1^2}{m_2e^{-w_2}+m_3e^{-w_3}}+m_2e^{w_2}+m_3e^{w_3}\right)\right). \tag{A.9}$$

Writing $w_2 = w_+ + w,\ w_3 = w_+ - w$, this becomes

$$-i\sqrt{2}\,\delta_{l_1,l_2+l_3}\int dw\left(e^{i(l_2-l_3)w}m_1^{-il_1}(m_2e^{-w}+m_3e^{w})^{-1+il_1}\times\right.$$
$$\left.\times\,\delta\left(\frac{-m_1^2+m_2^2+m_3^2+m_2m_3(e^{2w}+e^{-2w})}{m_2e^{-w}+m_3e^{w}}\right)\right). \tag{A.10}$$

The w integral gives rise to a sum of

$$-i\sqrt{2}\,\delta_{l_1,l_2+l_3}\frac{e^{i(l_2-l_3)w}m_1^{-il_1}(m_2e^{-w}+m_3e^{w})^{il_1}}{|8m_2m_3\sinh(w)\cosh(w)|} \tag{A.11}$$

over the roots $w = \pm w_0$ $(w_0 \geq 0)$ of the argument of the delta function:

$$\sinh^2(w_0) = \frac{m_1^2-(m_2+m_3)^2}{4m_2m_3}. \tag{A.12}$$

We finally obtain the following expression for the three point function with $m_1 > m_2 + m_3$ (the three point function vanishes if $m_1 < m_2 + m_3$):

$$\frac{-i\,\delta_{l_1,l_2+l_3}\,(e^{i\phi}+e^{i\phi'})}{\sqrt{2}\sqrt{m_1^2-(m_2+m_3)^2}\,\sqrt{m_1^2-(m_2-m_3)^2}}\,. \tag{A.13}$$

Here $e^{i\phi}$ and $e^{i\phi'}$ are phases:

$$\begin{aligned} e^{i\phi} &= e^{i(l_2-l_3)w_0}\left(\frac{m_2e^{-w_0}+m_3e^{w_0}}{m_1}\right)^{il_1};\\ e^{i\phi'} &= e^{-i(l_2-l_3)w_0}\left(\frac{m_2e^{w_0}+m_3e^{-w_0}}{m_1}\right)^{il_1}. \end{aligned} \tag{A.14}$$

For later convenience, let us note that when $m_1 \approx m_2 + m_3$, the three-point function can be simplified, because in this limit, $w_0 \approx 0$:

$$\langle\psi_1^*\psi_2\psi_3\rangle \approx \frac{-i\,\delta_{l_1,l_2+l_3}}{2\sqrt{m_1m_2m_3}\sqrt{m_1-m_2-m_3}}\,. \tag{A.15}$$

To see where this non-analyticity in the masses comes from, it is useful to analyze the three point function using the explicit form of the wavefunctions in terms of Hankel functions. We have

$$\begin{aligned} \langle\psi_1^*\psi_2\psi_3\rangle &= \frac{1}{16\sqrt{2}}e^{(l_1+l_2+l_3)\frac{\pi}{2}}\times\\ &\times\int dX^+dX^-\left(\frac{X^-}{X^+}\right)^{\frac{i}{2}(l_2+l_3-l_1)}H^{(1)}_{-il_1}(\tilde z_1)^*H^{(1)}_{-il_2}(\tilde z_2)H^{(1)}_{-il_3}(\tilde z_3) \end{aligned} \tag{A.16}$$

where we have used (2.10). (This expression is accurate in the future Milne wedge.)

Let us study the contribution from the asymptotic regions in the Milne wedges, as we did for the two point function in the previous subsection. Replacing the Hankel function by its asymptotics in the future Milne wedge, the contribution of this wedge to (A.16) becomes

$$\frac{e^{-\pi i/4}}{8\pi^{\frac{3}{2}}\sqrt{m_1m_2m_3}}\int dx e^{-i(l_2+l_3-l_1)x}\int\frac{dt}{\sqrt{t}}e^{-i(m_1-m_2-m_3)t}\,. \tag{A.17}$$

The t integral can be performed by rotating the integration contour, clockwise or counter-clockwise depending on the sign of $m_1-m_2-m_3$. If $m_1-m_2-m_3>0$, the past Milne wedge gives an identical contribution; the result of adding both is

$$\frac{-i\,\delta_{l_1,l_2+l_3}}{2\sqrt{m_1m_2m_3}\,\sqrt{m_1-m_2-m_3}}\,. \tag{A.18}$$

This reproduces (A.15), showing that the non-analyticity is associated with the asymptotic behavior of the wavefunctions in the Milne wedges. In the case $m_1-m_2-m_3<0$, the contributions of both Milne wedges cancel each other, which is consistent with the vanishing of the three point function in this case.

B. Operator product expansion and infrared divergences

In this appendix, we compute the OPE of the operators (2.9), and indicate how it can be used to explain infrared divergences in four point functions.

We have, using (2.10),

$$\psi_{m^2,l}=\frac{1}{2\sqrt{2}\pi i}\int_{-\infty}^{\infty}dw e^{i(\frac{m}{\sqrt{2}}X^-e^{-w}+\frac{m}{\sqrt{2}}X^+e^{w}+lw)}\,. \tag{B.1}$$

The worldsheet scaling dimension of the operator $\psi_{m^2,l}$ is $(-m^2/4,-m^2/4)$ (we have set $\alpha'=1$). Consider the OPE

$$\psi_{m_1^2,l_1}(z_1)\psi_{m_2^2,l_2}(z_2)\,. \tag{B.2}$$

We can perform the OPE of the plane waves in (B.1) and then integrate over the w's. The leading term in the OPE is

$$\begin{aligned}\psi_{m_1^2,l_1}(z_1)\psi_{m_2^2,l_2}(z_2)\;=\;&-\frac{1}{8\pi^2}\int_{-\infty}^{\infty}dw_1dw_2|z_1-z_2|^{-\frac{m_1m_2}{2}(e^{(w_1-w_2)}+e^{(w_2-w_1)})}e^{i(l_1w_1+l_2w_2)}\times\\&\times e^{\frac{i}{\sqrt{2}}(m_1e^{-w_1}+m_2e^{-w_2})X^-+\frac{i}{\sqrt{2}}(m_1e^{w_1}+m_2e^{w_2})X^+}\,.\end{aligned} \tag{B.3}$$

We would like to write the r.h.s. as

$$\int d(m^2)|z_1-z_2|^{(m_1^2+m_2^2-m^2)/2}C(m_1^2,m_2^2;m^2)\psi_{m^2;l_1+l_2}(z_2)\,. \tag{B.4}$$

Comparing (B.3) to (B.1), we see that

$$m^2=(m_1e^{-w_1}+m_2e^{-w_2})(m_1e^{w_1}+m_2e^{w_2})=m_1^2+m_2^2+2m_1m_2\cosh(w_1-w_2)\,. \tag{B.5}$$

Note that $m^2 \geq (m_1 + m_2)^2$ for non-negative m_1, m_2 (which we are assuming). The wavefunction (B.4) has

$$
\begin{aligned}
p^+ &= \frac{1}{\sqrt{2}}(m_1 e^{-w_1} + m_2 e^{-w_2}) \equiv \frac{m}{\sqrt{2}} e^{-w}\,; \\
p^- &= \frac{1}{\sqrt{2}}(m_1 e^{w_1} + m_2 e^{w_2}) \equiv \frac{m}{\sqrt{2}} e^{w}\,.
\end{aligned}
\tag{B.6}
$$

One can write the integral $\int dw_1 dw_2$ as $\int d(m^2) dw F(m^2, w)$ where F is the absolute value of the jacobian of the transformation (B.6). This jacobian is obtained by using (B.6) to express $w_{1,2}$ as functions of m, w, and then computing the partial derivatives:

$$
F = \frac{1}{\sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}}\,. \tag{B.7}
$$

The square root is precisely the product of the square roots in the denominator of the three point function (A.13), so it exhibits the same non-analyticity. In appendix A, we argued that this non-analyticity is an infrared effect. For each m, w there are two solutions for $w_{1,2}$, and they have to be summed over. The leading term in the OPE reads

$$
\begin{aligned}
\psi_1(z_1)\psi_2(z_2) \sim & -\frac{1}{8\pi^2} \int_{(m_1+m_2)^2}^{\infty} d(m^2) |z_1 - z_2|^{(m_1^2 + m_2^2 - m^2)/2} \times \\
& \times \frac{1}{\sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}} \times \\
& \times \left\{ \left[\frac{m^2 + m_1^2 - m_2^2 - \sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}}{2mm_1} \right]^{il_1} \times \right. \\
& \times \left[\frac{m^2 - m_1^2 + m_2^2 + \sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}}{2mm_2} \right]^{il_2} + \\
& + \left[\frac{m^2 + m_1^2 - m_2^2 + \sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}}{2mm_1} \right]^{il_1} \times \\
& \times \left. \left[\frac{m^2 - m_1^2 + m_2^2 - \sqrt{m^4 + m_1^4 + m_2^4 - 2m^2 m_1^2 - 2m^2 m_2^2 - 2m_1^2 m_2^2}}{2mm_2} \right]^{il_2} \right\} \times \\
& \times \int dw e^{i(l_1 + l_2)w} e^{i(\frac{m}{\sqrt{2}} X^- e^{-w} + \frac{m}{\sqrt{2}} X^+ e^{w})}\,.
\end{aligned}
\tag{B.8}
$$

In what follows, we will need the OPE

$$
\psi_{m_1^2, l_1}(z_1)(\psi_{m_2^2, l_2}(z_2))^*\,. \tag{B.9}
$$

This can be easily obtained in the same way as above. The right hand side of the OPE will now contain all $\psi_{m^2; l_1 - l_2}$ with $m^2 < (m_1 - m_2)^2$. The jacobian factor (B.7) is unchanged; it blows up when $m^2 \approx (m_1 - m_2)^2$.

Now consider the four point function (3.11) in the kinematical regime $m_1 = m_4$, $m_2 = m_3$. It exhibits a divergence from the region of the integral where $B \approx -A \approx \epsilon$, with ϵ

small and positive (we will be considering the limit $\epsilon \to 0$). In this region, v_2, v_3 and v_4 are all close to 1, and the jacobian (3.10) goes like $1/\epsilon$. This leads to a $\int d\epsilon/\epsilon$ divergence (for any transverse momenta). In fact, the jacobian (3.10) diverges whenever $v_2 = v_3$, which is possible whenever $(m_2 - m_3)^2 \leq (m_1 - m_4)^2$. However, the jacobian will only go like $1/\epsilon$ if this inequality is saturated, otherwise it only scales like $1/\sqrt{\epsilon}$.

This divergence is an infrared effect, and can be understood using the OPE we have just described. Assume for simplicity $m_2 \geq m_3$, and consider a process where particle 2 turns into particle 3, thereby emitting an intermediate (u-channel) particle with $m^2 = (m_2 - m_3)^2 - \epsilon$. The OPE gives a factor $1/\sqrt{\epsilon}$ from (B.7). The intermediate particle is now absorbed by particle 1, which turns into particle 4. From the OPE, this process is possible if $(m_2 - m_3)^2 \leq (m_1 - m_4)^2$. Now there are two possibilities: either there is a strict inequality, in which case the relevant coefficient in the second OPE is ϵ-independent; or the inequality is saturated, in which case the second OPE coefficient also scales like $1/\sqrt{\epsilon}$, so that the four point function has a $\int d\epsilon/\epsilon$ divergence.

C. Comparing the string theory and gravity computations

In this appendix we complete the computation in section 4 of the four-point function in dilaton-gravity, and compare it with the string theory result derived in section 3. In section 4, we computed the contribution of one of the four regions of the spacetime $\mathbb{R}^{1,1}/\mathbb{Z}$. In order to obtain the contribution of the other three regions, we first need to find useful expressions for the wavefunctions (2.9) in those regions.

C.1 The wavefunctions in the four regions

We need expressions for the behavior near the singularity of the wavefunctions (2.9). For the future Milne wedge, the result is given in (2.19). This can be extended to the other regions using the fact that (2.9) is built from purely negative frequency modes in Minkowski space, which are analytic on the lower half-planes of the complexified horizons. However, in this appendix we will derive the explicit expressions directly in the four regions of spacetime. We label the regions of $\mathbb{R}^{1,1}/\mathbb{Z}$ according to the sign of X^+, X^- as follows:

I $X^+, X^- < 0$;

II $X^+, X^- > 0$;

III $X^+ > 0, X^- < 0$;

IV $X^+ < 0, X^- > 0$.

The wavefunction in all regions is (2.9)

$$\psi(X^+, X^-) e^{i\vec{p}\cdot\vec{X}} = \frac{1}{2\sqrt{2}\pi i} \int dw e^{i(p^+ X^- e^{-w} + p^- X^+ e^{w} + lw)} e^{i\vec{p}\cdot\vec{X}} , \tag{C.1}$$

with $2p^+p^- = m^2$. As before, we will assume that all the m's and p's are positive, and set $p^+ = p^- = m/\sqrt{2}$.

We will use the following integral representations of the Hankel function:

$$
\begin{aligned}
H_\nu^{(1)}(z) &= \frac{1}{\pi i} e^{-\frac{i}{2}\pi\nu} \int_{-\infty}^{\infty} dt\, e^{iz\cosh(t)-\nu t} \qquad (z>0)\,; \\
H_\nu^{(1)}(xz) &= -\frac{i}{\pi} e^{-\frac{i}{2}\nu\pi} z^\nu \int_0^{\infty} dt\, e^{\frac{i}{2}x(t+z^2/t)} t^{-\nu-1}\,, \qquad (z=i,\ x>0)\,.
\end{aligned}
\tag{C.2}
$$

We also need the behavior of the wavefunctions ψ_l for small X^+, X^- in all the regions. It is useful to remember the leading behavior of the Hankel function $H_{il}^{(1)}(z)$ for small values of the argument z:

$$
H_{il}^{(1)}(z) = J_{il}(z) + iN_{il}(z) \sim \frac{1}{\sinh(l\pi)} \left[\frac{e^{l\pi}}{\Gamma(1+il)} \left(\frac{z}{2}\right)^{il} - \frac{1}{\Gamma(1-il)} \left(\frac{z}{2}\right)^{-il} \right]. \tag{C.3}
$$

Region I: $(X^+, X^-) < 0$. Defining $e^\beta = \sqrt{\frac{p^+X^-}{p^-X^+}}$ we write the wave function as

$$
\begin{aligned}
\psi_l &= \frac{1}{2\sqrt{2}\pi i} \int dw e^{-i\sqrt{p^+p^-X^+X^-}(e^{-w+\beta}+e^{w-\beta})+ilw} \\
&= \frac{1}{2\sqrt{2}\pi i} \left(\frac{p^+X^-}{p^-X^+}\right)^{il/2} \int dw e^{-i\sqrt{p^+p^-X^+X^-}(e^{-w}+e^{w})+ilw} \\
&= \frac{1}{2\sqrt{2}\pi i} \left(\frac{p^+X^-}{p^-X^+}\right)^{il/2} \left(\int dw e^{i\sqrt{p^+p^-X^+X^-}(e^{w}+e^{-w})-ilw} \right)^* \\
&= \frac{1}{2\sqrt{2}\pi i} \left(\frac{p^+X^-}{p^-X^+}\right)^{il/2} \left(e^{(\pi i/2)(il)} \pi i H_{il}(2\sqrt{p^+p^-X^+X^-}) \right)^* \\
&= \frac{-e^{-\pi l/2}}{2\sqrt{2}} \left(\frac{X^-}{X^+}\right)^{il/2} \left(H_{il}(2\sqrt{p^+p^-X^+X^-}) \right)^*,
\end{aligned}
\tag{C.4}
$$

where we have used $p^+ = p^-$ in the last line. The behavior near the origin is

$$
\psi_l \sim -\frac{1}{2\sqrt{2}} \frac{1}{\sinh(l\pi)} \left[\frac{e^{\pi l/2}}{\Gamma(1-il)} \left(-\frac{mX^+}{\sqrt{2}}\right)^{-il} - \frac{e^{-\pi l/2}}{\Gamma(1+il)} \left(-\frac{mX^-}{\sqrt{2}}\right)^{il} \right]. \tag{C.5}
$$

Region II: $(X^+, X^-) > 0$. Defining $e^\beta = \sqrt{\frac{p^+X^-}{p^-X^+}}$ we write the wave function here as

$$
\begin{aligned}
\psi_l &= \frac{1}{2\sqrt{2}\pi i} \int dw e^{i\sqrt{p^+p^-X^+X^-}(e^{-w+\beta}+e^{w-\beta})+ilw} \\
&= \frac{1}{2\sqrt{2}\pi i} \left(\frac{p^+X^-}{p^-X^+}\right)^{il/2} \int dw e^{i\sqrt{p^+p^-X^+X^-}(e^{-w}+e^{w})+ilw} \\
&= \frac{1}{2\sqrt{2}\pi i} \left(\frac{p^+}{p^-}\right)^{il/2} \left(\frac{X^-}{X^+}\right)^{il/2} e^{\frac{\pi i}{2}(-il)} \pi i H_{-il}(2\sqrt{p^+p^-X^+X^-}) \\
&= \frac{e^{\frac{\pi l}{2}}}{2\sqrt{2}} \left(\frac{X^-}{X^+}\right)^{il/2} H_{-il}\left(\sqrt{2m^2X^+X^-}\right).
\end{aligned}
\tag{C.6}
$$

The leading behavior of the wavefunction near the origin is

$$
\psi_l \sim \frac{1}{2\sqrt{2}} \frac{-1}{\sinh(\pi l)} \left[\frac{e^{-\pi l/2}}{\Gamma(1-il)} \left(\frac{mX^+}{\sqrt{2}}\right)^{-il} - \frac{e^{\pi l/2}}{\Gamma(1+il)} \left(\frac{mX^-}{\sqrt{2}}\right)^{il} \right]. \tag{C.7}
$$

This reproduces (2.19).

Region III: $(X^+, -X^-) > 0$. We define $e^\beta = \sqrt{-p^+X^-/p^-X^+}$ and write the leading term of the wave function as

$$\begin{aligned}
\psi_l &= \frac{1}{2\sqrt{2}\pi i}\left(\frac{-p^+X^-}{p^-X^+}\right)^{il/2} \int dw e^{i\sqrt{-p^+p^-X^+X^-}(-e^{-w+\beta}+e^{w-\beta})+il(w-\beta)} \\
&\sim \frac{1}{2\sqrt{2}\pi i}\left(\frac{-X^-}{X^+}\right)^{il/2} (\pi i) e^{\pi l/2} (i)^{il} \frac{-1}{\sinh(l\pi)} \\
&\times \left[\frac{e^{-l\pi}}{\Gamma(1-il)}\left(\frac{im\sqrt{X^+(-X^-)}}{\sqrt{2}}\right)^{-il} - \frac{1}{\Gamma(1+il)}\left(\frac{im\sqrt{X^+(-X^-)}}{\sqrt{2}}\right)^{il}\right] \\
&= \frac{1}{2\sqrt{2}}\frac{-1}{\sinh(l\pi)}\left[\frac{e^{-l\pi/2}}{\Gamma(1-il)}\left(\frac{m}{\sqrt{2}}\right)^{-il}(X^+)^{-il} - \frac{e^{-l\pi/2}}{\Gamma(1+il)}\left(\frac{m}{\sqrt{2}}\right)^{il}(-X^-)^{il}\right]. \qquad \text{(C.8)}
\end{aligned}$$

Region IV $(-X^+, X^-) > 0$. The behavior of the wave function near the origin is:

$$\psi_l \sim -\frac{1}{2\sqrt{2}}\frac{1}{\sinh(\pi l)}\left[\frac{e^{\pi l/2}}{\Gamma(1-il)}\left(\frac{m}{\sqrt{2}}\right)^{-il}(-X^+)^{-il} - \frac{e^{\pi l/2}}{\Gamma(1+il)}\left(\frac{m}{\sqrt{2}}\right)^{il}(X^-)^{il}\right]. \qquad \text{(C.9)}$$

C.2 The four point function

We are now ready to compute the four-point function in gravity and compare it to the string result. The four-point function, at least in the kinematic regime of section 4, is (4.4)

$$\begin{aligned}
\langle\psi_3^*\psi_4^*\psi_1\psi_2\rangle &= -(2\pi)^{24}\delta^{24}\left(\sum \epsilon_i\vec{p}_i\right) 2\pi \times \\
&\times \int dX^+dX^- \partial_-\psi_{m_1,l_1}\partial_-\psi^*_{m_3,l_3}\frac{1}{\partial^2}\partial_+\psi_{m_2,l_2}\partial_+\psi^*_{m_4,l_4}, \qquad \text{(C.10)}
\end{aligned}$$

where

$$\frac{1}{\partial^2} = -\frac{1}{(\vec{p}_2-\vec{p}_4)^2}. \qquad \text{(C.11)}$$

The contribution to the four-point function from the region near the singularity may be evaluated by simply inserting the expressions for the leading behavior of the wavefunction from the various regions (C.5)–(C.9). Define

$$S(m_i,l_i,p_i) = \frac{(2\pi)^{24}\delta^{24}(\sum\epsilon_i\vec{p}_i)2\pi l_1l_2l_3l_4\left(\frac{m_1}{\sqrt{2}}\right)^{il_1}\left(\frac{m_2}{\sqrt{2}}\right)^{-il_2}\left(\frac{m_3}{\sqrt{2}}\right)^{-il_3}\left(\frac{m_4}{\sqrt{2}}\right)^{il_4}}{64(\prod\sinh(\pi l_i))\Gamma(1+il_1)\Gamma(1-il_2)\Gamma(1-il_3)\Gamma(1+il_4)(\vec{p}_1-\vec{p}_3)^2}, \qquad \text{(C.12)}$$

in terms of which the contribution to the four-point function from the various regions are

Region I:

$$S(m_i,l_i,p_i)e^{\pi(-l_1+l_2-l_3+l_4)/2}\int_{-\infty}^0 dX^+\int_{-\infty}^0 dX^-(-X^+)^{i(l_4-l_2)-2}(-X^-)^{i(l_1-l_3)-2}; \qquad \text{(C.13)}$$

Region II:

$$S(m_i,l_i,p_i)e^{\pi(l_1-l_2+l_3-l_4)/2}\int_0^\infty dX^+\int_0^\infty dX^-(X^+)^{i(l_4-l_2)-2}(X^-)^{i(l_1-l_3)-2}; \qquad \text{(C.14)}$$

Region III:

$$S(m_i,l_i,p_i)e^{\pi(-l_1-l_2-l_3-l_4)/2}\int_0^{\infty} dX^+\int_{-\infty}^{0} dX^-\,(X^+)^{i(l_4-l_2)-2}(-X^-)^{i(l_1-l_3)-2}\,; \qquad (C.15)$$

Region IV:

$$S(m_i,l_i,p_i)e^{\pi(l_1+l_2+l_3+l_4)/2}\int_{-\infty}^{0} dX^+\int_0^{\infty} dX^-(-X^+)^{i(l_4-l_2)-2}(X^-)^{i(l_1-l_3)-2}\,. \qquad (C.16)$$

The complete four-point function is therefore

$$\begin{aligned}\langle\psi_3^*\psi_4^*\psi_1\psi_2\rangle = &\frac{(2\pi)^{24}\delta^{24}(\sum\epsilon_i\vec{p}_i)2\pi l_1l_2l_3l_4\left(\frac{m_1}{\sqrt{2}}\right)^{il_1}\left(\frac{m_2}{\sqrt{2}}\right)^{-il_2}\left(\frac{m_3}{\sqrt{2}}\right)^{-il_3}\left(\frac{m_4}{\sqrt{2}}\right)^{il_4}}{64(\prod\sinh(\pi l_i))\Gamma(1+il_1)\Gamma(1-il_2)\Gamma(1-il_3)\Gamma(1+il_4)(\vec{p}_1-\vec{p}_3)^2}\times\\ &\times\Big(e^{\pi(-l_1+l_2-l_3+l_4)/2}+e^{\pi(l_1-l_2+l_3-l_4)/2}+e^{\pi(-l_1-l_2-l_3-l_4)/2}+e^{\pi(l_1+l_2+l_3+l_4)/2}\Big)\times\\ &\times\int_0^{\infty}dX^+\int_0^{\infty}dX^-(X^+)^{i(l_4-l_2)-2}(X^-)^{i(l_1-l_3)-2}\,. \end{aligned} \qquad (C.17)$$

Using $X^\pm=\frac{1}{\sqrt{2}}e^{\eta\pm x}$, we can write the integral in the above expression as

$$\begin{aligned}&\int_0^{\infty}dX^+\int_0^{\infty}dX^-(X^+)^{i(l_4-l_2)-2}(X^-)^{i(l_1-l_3)-2}=\\ &\qquad=2\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}dx\left(\frac{e^{\eta}}{\sqrt{2}}\right)^{i(l_1+l_4-l_2-l_3)-2}e^{ix(l_3+l_4-l_1-l_2)}.\end{aligned} \qquad (C.18)$$

Doing the x integral, and defining $v=2e^{-2\eta}$, we can simplify this to

$$2\pi\delta(l_1+l_2-l_3-l_4)\int_0^{\infty}dv\,v^{i(l_2+l_3-l_1-l_4)/2}\,. \qquad (C.19)$$

Finally, writing $v=\frac{1}{2}m_1m_4u$, the four point function becomes

$$\begin{aligned}\langle\psi_3^*\psi_4^*\psi_1\psi_2\rangle \;=\;& \frac{(2\pi)^{24}}{16}\delta^{24}\left(\sum\epsilon_i\vec{p}_i\right)\delta\left(\sum\epsilon_il_i\right)\frac{m_1m_4}{(\vec{p}_1-\vec{p}_3)^2}\left(\frac{m_4}{m_2}\right)^{il_2}\left(\frac{m_3}{m_1}\right)^{-il_3}\int_0^{\infty}du\,u^{i(l_2-l_4)}\times\\ &\times\frac{\pi^2l_1l_2l_3l_4\left(e^{\frac{\pi(-l_1+l_2-l_3+l_4)}{2}}+e^{\frac{\pi(l_1-l_2+l_3-l_4)}{2}}+e^{\frac{\pi(-l_1-l_2-l_3-l_4)}{2}}+e^{\frac{\pi(l_1+l_2+l_3+l_4)}{2}}\right)}{2(\prod\sinh(\pi l_i))\Gamma(1+il_1)\Gamma(1-il_2)\Gamma(1-il_3)\Gamma(1+il_4)}\,.\end{aligned} \qquad (C.20)$$

This expression is identical to the $\alpha'\to 0$ limit of the string theory result (3.16) provided we rescale

$$u=A(l_i)v_4\,, \qquad (C.21)$$

where the coefficient $A(l_i)$ is independent of the masses m_i.

As a consistency check, this rescaling does nothing in the (unphysical) case $1+i(l_2-l_4)=1+i(l_3-l_1)=0$. Therefore, we expect that for these values of l_i, the second line of (C.20) should not depend on l_i. In fact, using standard Gamma function identities, it is easy to show that

$$
\begin{aligned}
&\frac{\pi^2 l_1 l_2 l_3 l_4 (e^{\pi(-l_1+l_2-l_3+l_4)/2} + e^{\pi(l_1-l_2+l_3-l_4)/2} + e^{\pi(-l_1-l_2-l_3-l_4)/2} + e^{\pi(l_1+l_2+l_3+l_4)/2})}{2(\prod \sinh(\pi l_i)) = \Gamma(1+il_1)\Gamma(1-il_2)\Gamma(1-il_3)\Gamma(1+il_4)} \\
&= \frac{\pi^2(-i+l_3)l_2 l_3(-i+l_2)\left(e^{\pi(l_1-l_4)} + e^{-\pi(l_1-l_4)} - e^{-\pi(l_1+l_4)} - e^{\pi(l_1+l_4)}\right)}{2\Pi_i \sinh(\pi l_i)\Gamma(2+il_3)\Gamma(1-il_3)\Gamma(1-il_2)\Gamma(2+il_2)} \\
&= -\frac{\pi^2(1+il_3)l_2 l_3(1+il_2)\left(e^{\pi(l_1-l_4)} + e^{-\pi(l_1-l_4)} - e^{-\pi(l_1+l_4)} - e^{\pi(l_1+l_4)}\right)}{2\Pi_i \sinh(\pi l_i)\Gamma(2+il_3)\Gamma(1-il_3)\Gamma(1-il_2)\Gamma(2+il_2)} \\
&= -\frac{\pi^2 l_2 l_3\left(e^{\pi(l_1-l_4)} + e^{-\pi(l_1-l_4)} - e^{-\pi(l_1+l_4)} - e^{\pi(l_1+l_4)}\right)}{2\Pi_i \sinh(\pi l_i)\Gamma(1+il_3)\Gamma(1-il_3)\Gamma(1-il_2)\Gamma(1+il_2)} \\
&= \frac{e^{\pi(l_1+l_4)} + e^{-\pi(l_1+l_4)} - e^{\pi(l_1-l_4)} - e^{-\pi(l_1-l_4)}}{2\sinh(\pi l_4)\sinh(\pi l_1)} \\
&= 2\,. \qquad\qquad (C.22)
\end{aligned}
$$

References

[1] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt and N. Turok, *From big crunch to big bang*, *Phys. Rev.* **D 65** (2002) 086007 [hep-th/0108187].

[2] N. Seiberg, *From big crunch to big bang-is it possible?*, hep-th/0201039.

[3] V. Balasubramanian, S.F. Hassan, E. Keski-Vakkuri and A. Naqvi, *A space-time orbifold: a toy model for a cosmological singularity*, *Phys. Rev.* **D 67** (2003) 026003 [hep-th/0202187].

[4] L. Cornalba and M.S. Costa, *A new cosmological scenario in string theory*, *Phys. Rev.* **D 66** (2002) 066001 [hep-th/0203031].

[5] N.A. Nekrasov, *Milne universe, tachyons and quantum group*, hep-th/0203112.

[6] J. Simon, *The geometry of null rotation identifications*, *J. High Energy Phys.* **06** (2002) 001 [hep-th/0203201].

[7] A.J. Tolley and N. Turok, *Quantum fields in a big crunch/big bang spacetime*, *Phys. Rev.* **D 66** (2002) 106005 [hep-th/0204091].

[8] H. Liu, G. Moore and N. Seiberg, *Strings in a time-dependent orbifold*, *J. High Energy Phys.* **06** (2002) 045 [hep-th/0204168].

[9] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *From big bang to big crunch and beyond*, *J. High Energy Phys.* **06** (2002) 017 [hep-th/0204189].

[10] L. Cornalba, M.S. Costa and C. Kounnas, *A resolution of the cosmological singularity with orientifolds*, *Nucl. Phys.* **B 637** (2002) 378 [hep-th/0204261].

[11] B. Craps, D. Kutasov and G. Rajesh, *String propagation in the presence of cosmological singularities*, *J. High Energy Phys.* **06** (2002) 053 [hep-th/0205101].

[12] S. Kachru and L. McAllister, *Bouncing brane cosmologies from warped string compactifications*, hep-th/0205209.

[13] A. Lawrence, *On the instability of 3d null singularities*, *J. High Energy Phys.* **11** (2002) 019 [hep-th/0205288].

JHEP03(2003)031

[14] C. Gordon and N. Turok, *Cosmological perturbations through a general relativistic bounce*, hep-th/0206138.

[15] E.J. Martinec and W. McElgin, *Exciting AdS orbifolds*, *J. High Energy Phys.* **10** (2002) 050 [hep-th/0206175].

[16] H. Liu, G. Moore and N. Seiberg, *Strings in time-dependent orbifolds*, *J. High Energy Phys.* **10** (2002) 031 [hep-th/0206182].

[17] M. Fabinger and J. McGreevy, *On smooth time-dependent orbifolds and null singularities*, hep-th/0206196.

[18] G.T. Horowitz and J. Polchinski, *Instability of spacelike and null orbifold singularities*, *Phys. Rev.* **D 66** (2002) 103512 [hep-th/0206228].

[19] A. Buchel, P. Langfelder and J. Walcher, *On time-dependent backgrounds in supergravity and string theory*, *Phys. Rev.* **D 67** (2003) 024011 [hep-th/0207214].

[20] S. Hemming, E. Keski-Vakkuri and P. Kraus, *Strings in the extended btz spacetime*, *J. High Energy Phys.* **10** (2002) 006 [hep-th/0208003].

[21] A. Hashimoto and S. Sethi, *Holography and string dynamics in time-dependent backgrounds*, *Phys. Rev. Lett.* **89** (2002) 261601 [hep-th/0208126].

[22] J. Simon, *Null orbifolds in AdS, time dependence and holography*, *J. High Energy Phys.* **10** (2002) 036 [hep-th/0208165].

[23] M. Alishahiha and S. Parvizi, *Branes in time-dependent backgrounds and AdS/CFT correspondence*, *J. High Energy Phys.* **10** (2002) 047 [hep-th/0208187].

[24] Y. Satoh and J. Troost, *Massless btz black holes in minisuperspace*, *J. High Energy Phys.* **11** (2002) 042 [hep-th/0209195].

[25] R.-G. Cai, J.-X. Lu and N. Ohta, *Ncos and D-branes in time-dependent backgrounds*, *Phys. Lett.* **B 551** (2003) 178 [hep-th/0210206].

[26] C. Bachas and C. Hull, *Null brane intersections*, *J. High Energy Phys.* **12** (2002) 035 [hep-th/0210269].

[27] K. Okuyama, *D-branes on the null-brane*, *J. High Energy Phys.* **02** (2003) 043 [hep-th/0211218].

[28] G. Papadopoulos, J.G. Russo and A.A. Tseytlin, *Solvable model of strings in a time-dependent plane-wave background*, *Class. and Quant. Grav.* **20** (2003) 969 [hep-th/0211289].

[29] G.T. Horowitz and A.R. Steif, *Singular string solutions with nonsingular initial data*, *Phys. Lett.* **B 258** (1991) 91.

[30] S.W. Hawking, *The chronology protection conjecture*, *Phys. Rev.* **D 46** (1992) 603.

[31] C.R. Nappi and E. Witten, *A closed, expanding universe in string theory*, *Phys. Lett.* **B 293** (1992) 309 [hep-th/9206078].

[32] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[33] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the* $(2+1)$ *black hole*, *Phys. Rev.* **D 48** (1993) 1506 [gr-qc/9302012].

[34] P. Kraus, H. Ooguri and S. Shenker, *Inside the horizon with AdS/CFT*, hep-th/0212277.

[35] A. Sen, *Rolling tachyon*, *J. High Energy Phys.* **04** (2002) 048 [hep-th/0203211].

[36] A. Sen, *Field theory of tachyon matter*, *Mod. Phys. Lett.* **A 17** (2002) 1797 [hep-th/0204143].

[37] N.D. Birrell and P.C. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge 1982.

[38] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Academic Press 1994.

[39] S.M. Carroll, *Lecture notes on general relativity*, gr-qc/9712019.

[40] C. Kounnas and D. Lüst, *Cosmological string backgrounds from gauged WZW models*, *Phys. Lett.* **B 289** (1992) 56 [hep-th/9205046].

[41] D. Lüst, *Cosmological string backgrounds*, hep-th/9303175.

[42] J.M. Maldacena, *Eternal black holes in anti-de-Sitter*, hep-th/0106112.

[43] A. Sen, *Tachyon matter*, *J. High Energy Phys.* **07** (2002) 065 [hep-th/0203265].

[44] A. Strominger, *Open string creation by S-branes*, hep-th/0209090.

[45] A.A. Gerasimov and S.L. Shatashvili, *On exact tachyon potential in open string field theory*, *J. High Energy Phys.* **10** (2000) 034 [hep-th/0009103].

[46] D. Kutasov, M. Marino and G.W. Moore, *Some exact results on tachyon condensation in string field theory*, *J. High Energy Phys.* **10** (2000) 045 [hep-th/0009148].

[47] D. Kutasov, M. Marino and G.W. Moore, *Remarks on tachyon condensation in superstring field theory*, hep-th/0010108.

[48] N.D. Lambert and I. Sachs, *Tachyon dynamics and the effective action approximation*, *Phys. Rev.* **D 67** (2003) 026005 [hep-th/0208217].

[49] G. Gibbons, K. Hashimoto and P. Yi, *Tachyon condensates, carrollian contraction of Lorentz group and fundamental strings*, *J. High Energy Phys.* **09** (2002) 061 [hep-th/0209034].

[50] A. Sen, *Time and tachyon*, hep-th/0209122.

[51] G.N. Felder, L. Kofman and A. Starobinsky, *Caustics in tachyon matter and other Born-Infeld scalars*, *J. High Energy Phys.* **09** (2002) 026 [hep-th/0208019].

[52] K. Hashimoto, P.-M. Ho and J.E. Wang, *S-brane actions*, hep-th/0211090.

[53] A. Sen, *Time evolution in open string theory*, *J. High Energy Phys.* **10** (2002) 003 [hep-th/0207105].

[54] J.A. Harvey, D. Kutasov and E.J. Martinec, *On the relevance of tachyons*, hep-th/0003101.

JHEP03(2003)031